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Effects of Observer Dynamics on Motion Stability of Autonomous Vehicles

by

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TABLE OF SYMBOLS

a	dummy independent variable, or yaw rate coefficient in Nomoto's model
A	linearized system matrix
b	rudder angle in Nomoto's model
c	parameter for variance of gain and hydrodynamic coefficients
c_{crit}	bifurcation value of c
I_z	vehicle mass moment of inertia
K	cubic stability coefficient
K_ψ, K_r, K_y	controller gains
ℓ_ψ, ℓ_r, ℓ_y	observer gains
m	vehicle mass
N	yaw moment
PAH	Poincaré-Andronov-Hopf Bifurcation
r	yaw rate
R	polar coordinate of transformed reduced system
T	matrix of eigenvectors of A, or limit cycle period
v	sway velocity
X	state variables vector
x_G	body fixed coordinate of vehicle center of gravity
y	deviation of the commanded path

Y	sway force
z	stable variables vector in canonical form
z_1, z_2	critical variables of z
α_0, α_1, α_2	coefficients of desired characteristic equation
β	real part of critical pair of eigenvalues
β'	derivative of β with respect to c evaluated at c_{crit}
γ_0, γ_1, γ_2	coefficients of desired characteristic equation
δ	rudder angle control law
δ_0	linearized rudder angle control law
ϵ	critical difference $c - c_{crit}$
θ	polar coordinate of transformed reduced system
ψ	vehicle heading angle
ω	imaginary part of critical pair of eigenvalues
ω'	derivative of ω with respect to c evaluated at c_{crit}
ω_n	natural frequency
ω_{n0}	observer natural frequency

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I. INTRODUCTION

Accurate path control of surface ships and underwater vehicles along prescribed geographical paths is a basic problem that is becoming increasingly important, particularly as the missions of ocean vehicles become more complicated with strict requirements for performance. In order for a control law to be able to perform its mission in a realistic operational scenario it has to be robust enough so that it can maintain stability and accuracy of operations in the presence of modeling errors and environmental uncertainties. The robustness properties of the design are particularly important due to the unpredictable nature of the ocean environment and the changes in the hydrodynamic characteristics of the vehicle during turning, changes in the forward speed, or operations in proximity to other objects in the area. For these reasons, there exists a need for the analysis of the robustness characteristics of a particular control law design and the establishment of a rational operational envelope based on stability and performance criteria. Previous studies [Parsons and Cuong (1977)] showed that gain adaption is highly desirable due to changes in the linearized vehicle hydrodynamics with different operation conditions, such as depth under keel. The resulting adaptation scheme [Parsons and Cuong (1980)] required significant vehicle motion, which may be undesirable when operating in restricted waters, or in object recognition and localization tasks. Integral control techniques [Parsons and Cuong (1981)] proved quite effective, but neglected the nonlinear behavior of the vehicle, which becomes very important at low speeds and hover operations. Model based compensators exhibit robust behavior under conditions of parameter uncertainty, which is as good as the classical linear quadratic

regulators for linear output feedback systems [Healey (1992)]. Alternatively, sliding mode controllers exhibit very robust characteristics given an estimate of the parameter uncertainty and/or disturbances [Papoulias and Healey (1992)], [Yoerger and Slotine (1985)]. Sliding mode control, however, does not offer an infinitely robust design and it suffers from a series of bifurcation phenomena and loss of stability unless proper care is exercised [Papoulias (1991)].

In this work we analyze the problem of loss of stability of a marine vehicle under cross track error control in the presence of mathematical versus actual system mismatch. For demonstration purposes, variations in the heading angle control gain are studied. Previous studies [Oral (1993)] concentrated on system response assuming perfect and complete state measurement. Particular emphasis in this work is placed on analyzing the effects of observer design on system response after initial loss of stability of straight line motion. The main loss of stability cases analyzed here occur in the form of generic bifurcations to periodic solutions [Guckenheimer and Holmes (1983)]. We use center manifold reduction techniques and averaging in order to capture the stability properties of the resulting limit cycles [Chow and Mallet-Paret (1977)]. It is shown that the dynamics of the observer may have a significant effect on the computed gain margin of the control system depending on the particular basis used. All computations in this work are conducted for the NPS autonomous underwater vehicle [Bahrke (1992)] and all results are presented in standard dimensionless quantities with respect to vehicle length, 7.3 ft, and nominal forward speed, 2 ft/sec.

II. PROBLEM FORMULATION

A. EQUATIONS OF MOTION

The linear maneuvering equations of motion of a marine vehicle in the horizontal plane are written in dimensionless form as,

$$m(\dot{\nu} + r + x_G\dot{r}) = Y_{\dot{r}}\dot{r} + Y_{\dot{\nu}}\dot{\nu} + Y_r r + Y_\nu \nu + Y_\delta \delta \tag{2.1}$$

$$I_z\dot{r} + mx_G(\dot{\nu} + r) = N_{\dot{r}}\dot{r} + N_{\dot{\nu}}\dot{\nu} + N_r r + N_\nu \nu + N_\delta \delta \tag{2.2}$$

where all symbols are explained in the nomenclature. Equations (2.1) and (2.2) can be used to derive a second order transfer function between the rudder angle δ and yaw rate r. For low frequency maneuvering motions this second order equation can be approximated by expanding in Taylor series and keeping the first order terms only. The result is

$$\dot{r} = ar + b\delta \tag{2.3}$$

Equation (2.3), which is sometimes referred to as Nomoto's first order model, is particularly useful in control system design since no sway velocity feedback is necessary. This equation predicts linear variation of the steady state turning rate versus rudder angle. In reality, the r-δ curve has characteristics of softening spring mainly due to speed loss during turning. To account for this a modified version of equation (2.3) is used,

$$\dot{r} = ar + a_3r^3 + b\delta \tag{2.4}$$

where a_3 is usually determined from steady state results. Finally, the model is complete by the incorporation of the kinematic equations,

$$\dot{\Psi} = r \tag{2.5}$$

$$\dot{y} = \sin \Psi \tag{2.6}$$

where Ψ is the vehicle heading, and y is the cross track error off a desired straight line path.

B. COMPENSATOR DESIGN

In control theory it is known that the eigenvalues of the controller are not affected by the eigenvalues of the observer. This allows us to design the controller and observer separately which is known as the separation principle. The combination is called a compensator.

Equations (2.3), (2.5), and (2.6) govern the steering control of the model used in this section. The control law can be expressed as,

$$\delta = \delta_{sat} \tanh \left(\frac{\delta_0}{\delta_{sat}} \right) \tag{2.7}$$

where around the nominal state $\Psi = r = y = 0$ we have

$$\delta_0 = K_\Psi \Psi + K_r r + K_y y \tag{2.8}$$

δ is the rudder angle and K_Ψ, K_r, and K_y are the control gains of the system. The linear control law is δ_0. The rudder angle δ is defined by a hyperbolic tangent function to include the saturation to our problem as shown in Figure 2.1. Saturation occurs at δ_{sat}, which is the saturation limit generally taken as 0.4 rad.

The linearized form of equations of motions in the vicinity of $\Psi = r = y = 0$ are,

$$\dot{\Psi} = r \tag{2.9}$$

$$\dot{r} = ar + b\delta_0 \tag{2.10}$$

$$\dot{y} = \Psi \tag{2.11}$$

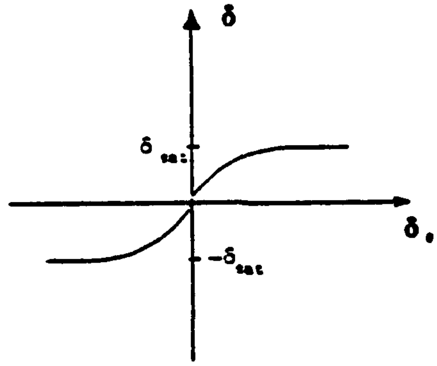

Figure 2.1: Saturation in δ.

These equations can be expressed in state space form as

$$\dot{X} = AX + Bu \tag{2.12}$$

where

$$X = \begin{bmatrix} \Psi \\ r \\ y \end{bmatrix}$$

is the state vector,

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & a & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

is the open loop dynamics matrix and

$$B = \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix}$$

is the control distribution vector.

The observer equations are

$$\dot{\hat{X}} = A\hat{X} + Bu + L(Y - C\hat{X}) \tag{2.13}$$

where $\hat{X}$ is the estimate of X, Y is the output of the system $Y = y$, and C is the sensor vector $C = [0\ 0\ 1]$.

The error in the estimate of X is defined by

$$\dot{\tilde{X}} = \dot{X} - \dot{\hat{X}} \tag{2.14}$$

Using equations (2.12), (2.13), and (2.14) we can obtain

$$\dot{\tilde{X}} = (A - LC)\tilde{X} \tag{2.15}$$

We can rewrite equations (2.9), (2.10), and (2.11) in the form of

$$\dot{X} = \begin{bmatrix} \dot{\Psi} \\ \dot{r} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & a & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Psi \\ r \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} \delta \tag{2.16}$$

and

$$Y = [0\ 0\ 1] \begin{bmatrix} \Psi \\ r \\ y \end{bmatrix} \tag{2.17}$$

The observer gains are,

$$L = \begin{bmatrix} \ell_\Psi \\ \ell_r \\ \ell_y \end{bmatrix} \tag{2.18}$$

After performing the matrix operations we obtain

$$\dot{\tilde{X}} = \begin{bmatrix} \dot{\tilde{\Psi}} \\ \dot{\tilde{r}} \\ \dot{\tilde{y}} \end{bmatrix} = \begin{bmatrix} 0 & 1 & -\ell_\Psi \\ 0 & a & -\ell_r \\ 1 & 0 & -\ell_y \end{bmatrix} \begin{bmatrix} \tilde{\Psi} \\ \tilde{r} \\ \tilde{y} \end{bmatrix} \tag{2.19}$$

Using equation (2.13) we can rewrite equation (2.8) as follows,

$$\delta_0 = K_\Psi(\Psi - \tilde{\Psi}) + K_r(r - \tilde{r}) + K_y(y - \tilde{y})$$

Finally, we can write our compensator equations in the form

$$\begin{bmatrix} \dot{X} \\ \dot{\tilde{X}} \end{bmatrix} = \begin{bmatrix} \dot{\Psi} \\ \dot{r} \\ \dot{y} \\ \dot{\tilde{\Psi}} \\ \dot{\tilde{r}} \\ \dot{\tilde{y}} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & bcK_\Psi & bK_r & bK_y \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -\ell_\Psi \\ 0 & 0 & 0 & 0 & a & -\ell_r \\ 0 & 0 & 0 & 1 & 0 & -\ell_y \end{bmatrix} \begin{bmatrix} \Psi \\ r \\ y \\ \tilde{\Psi} \\ \tilde{r} \\ \tilde{y} \end{bmatrix}$$

If we look at the matrix carefully we will see that it is in the form

$$\begin{bmatrix} \dot{X} \\ \dot{\tilde{X}} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} X \\ \tilde{X} \end{bmatrix}$$

which has the following characteristic equation,

$$\det[A - BK - sI] \det[A - LC - sI] = 0$$

This indicates that the dynamics of the observer are completely independent of the dynamics (eigenvalues) of the controller. Thus K and L can be designed separately.

C. CALCULATION OF CONTROL GAINS

A is the Jacobian matrix of the system

$$A = \begin{bmatrix} 0 & 1 & 0 \\ bK_\psi & a + bK_r & bK_y \\ 1 & 0 & 0 \end{bmatrix}$$

The characteristic equation of the matrix A is

$$\lambda^3 - (a + bK_r)\lambda^2 - bK_\psi\lambda - bK_y = 0$$

If the desired characteristic equation has the general form

$$\lambda^3 + \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0 = 0$$

the control gains can be found as

$$\begin{aligned} K_\psi &= -\frac{\alpha_1}{b} \\ K_r &= -\frac{\alpha_2 + a}{b} \\ K_y &= -\frac{\alpha_0}{b} \end{aligned}$$

The desired characteristic equation can be written with respect to the desired natural frequency and some optimum coefficients. The ITAE criterion for a third order equation is

$$s^3 + 1.75w_n s^2 + 2.15w_n^2 s + w_n^3 = 0$$

where w_n is the desired controller natural frequency.

Therefore the control gains can be calculated for a given natural frequency, as

$$\begin{aligned}\alpha_1 &= 2.15w_n^2\\ \alpha_2 &= 1.75w_n\\ \alpha_0 &= w_n^3\end{aligned}$$

D. CALCULATION OF OBSERVER GAINS

If we define A as the Jacobian matrix of the system

$$A = \begin{bmatrix} 0 & 1 & -\ell_\psi \\ 0 & a & -\ell_r \\ 1 & 0 & -\ell_y \end{bmatrix}$$

the characteristic equation of the matrix A is

$$\lambda^3 + (\ell_y - a) + (\ell_\psi - a\ell_y)\lambda + (\ell_r - a\ell_\psi) = 0$$

If the desired characteristic equation has the general form

$$\lambda^3 + \gamma_2\lambda^2 + \gamma_1\lambda + \gamma_0 = 0$$

the observer gains can be found as

$$\begin{aligned}\ell_y &= a + \gamma_2\\ \ell_\psi &= a\ell_y + \gamma_1\\ \ell_r &= a\ell_\psi + \gamma_0\end{aligned}$$

Applying the ITAE criteria, observer gains can be calculated for a given natural frequency as

$$\begin{aligned}\gamma_1 &= 2.15w_{n0}^2\\ \gamma_2 &= 1.75w_{n0}\\ \gamma_0 &= w_{n0}^3\end{aligned}$$

where w_{n0} is the observer natural frequency.

E. CHARACTERISTICS OF ITAE CRITERIA

In the calculations of gains we applied the ITAE criteria. If we look at Figure 2.2 for the step response of ITAE, we see that the response gets faster as the natural frequency increases. For example, the settling time is 10 normalized seconds, or 10 seconds for $w_n = 1$, 1 second for $w_n = 10$, and so on.

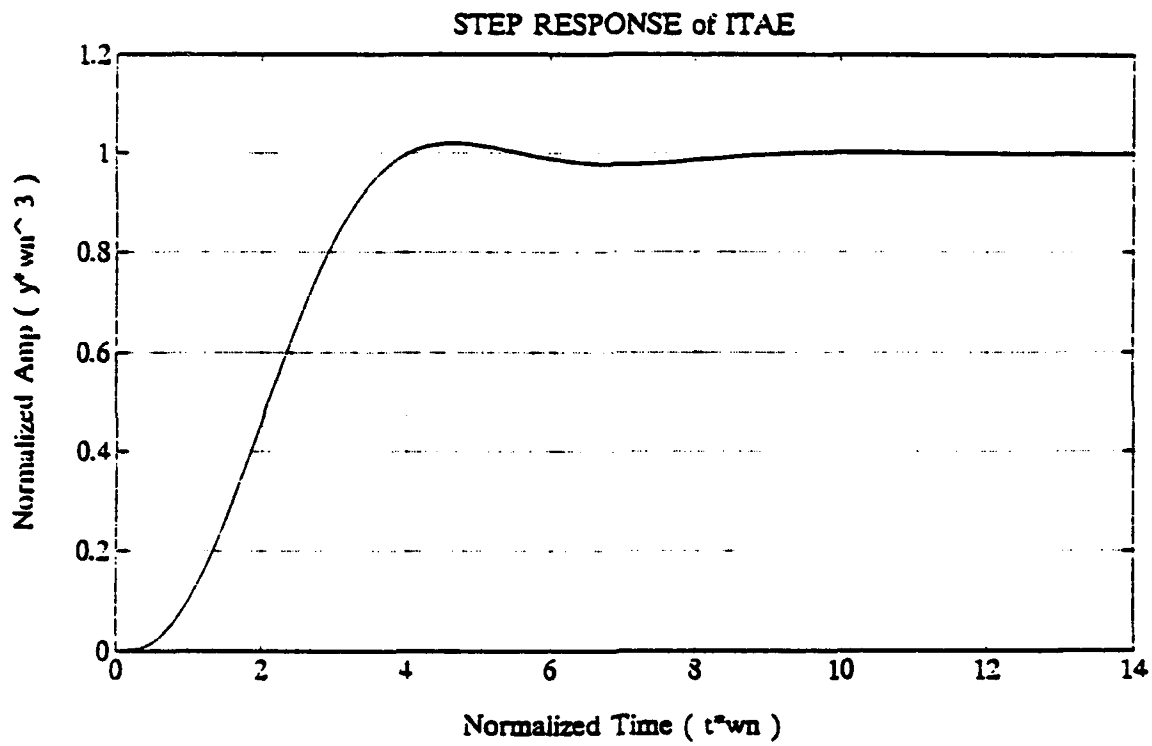

Figure 2.2: Step response of ITAE.

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III. HOPF BIFURCATION

A. INTRODUCTION

An important quantity in assessing the robustness of a particular control law design to parameter variations and unmodeled dynamics is the gain margin. This is defined as the extent to which changes can be inflicted on the system gain without loss in stability. to this end, we assume that the heading error gain K_ψ is multiplied by a constant C. By definition, of Hopf bifurcation occurs when a pair of complex conjugate eigenvalues cross into the right hand half-plane. When this occurs the system will deviate from a steady solution in an oscillatory manner. This deviation can be either supercritical or subcritical [Seydel (1988)]. As the parameter C crosses the critical value, one pair of complex conjugate eigenvalues of the linear system matrix crosses transversely the imaginary axis. Locally, as C approaches C_{crit}, the periodic solutions are located on the two dimensional Eucledian plane spanned by the eigenvectors of the Jacobian matrix of the system which corresponds to the critical pair of eigenvalues. In this chap· · stability properties of the periodic solutions are established. In order to establish those properties the main nonlinear terms that dominate the system are isolated. Center manifold theory is used to reduce the flow to a two dimensional manifold. The method of averaging is then applied to the reduced system.

The critical value of c for stability of straight line motion remains the same as [Oral (1993)], which is

$$C_{crit} = 0.2658$$

This is because the dynamics of the controller are independent from the dynamics of the observer as explained in Chapter II.

B. THIRD ORDER EXPANSIONS OF THE SYSTEM EQUATIONS

1. Perturbation in K_Ψ

In the previous chapter we worked on the linear system. Now we are going to introduce the nonlinear terms to our compensator. In this case the equations of motion are

$$\dot{\Psi} = r \tag{3.1}$$

$$\dot{r} = ar + a_3 r^3 + b\delta \tag{3.2}$$

$$\dot{y} = \sin\Psi \tag{3.3}$$

where

$$\delta = \delta_{sat} \cdot \tanh\left(\frac{\delta_0}{\delta_{sat}}\right) \tag{3.4}$$

$$\delta_0 = CK_\Psi(\Psi - \tilde{\Psi}) + K_r(r - \tilde{r}) + K_y(y - \tilde{y}) \tag{3.5}$$

or in compact form,

$$\dot{X} = f(x), \qquad X = \left[\Psi,\ r,\ y,\ \tilde{\Psi},\ \tilde{r},\ \tilde{y}\right]^T \tag{3.6}$$

This system can be written in the form

$$\dot{X} = AX + g(x) \tag{3.7}$$

A is the Jacobian matrix of $f(x)$ evaluated at $X = 0$, and $g(x)$ contains all nonlinear terms of Equations (3.1), (3.2), and (3.3). Taylor expansion of the nonlinear terms about the equilibrium, where we keep the first non-vanishing nonlinear coefficients

only, gives

$$\sin\Psi = \Psi - \frac{1}{6}\Psi^3 \tag{3.8}$$

$$\delta = \delta_0 - \frac{1}{3\delta_{sat}^2}\delta_0^3 \tag{3.9}$$

Substitution of Equations (3.8) and (3.9) into Equations (3.1), (3.2), and (3.3) gives us the A matrix in Equation (3.7) as follows,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ bcK_\Psi & a+bK_r & bK_y & -bcK_\Psi & -bK_r & -bK_y \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -\ell_\Psi \\ 0 & 0 & 0 & 0 & a & -\ell_r \\ 0 & 0 & 0 & 1 & 0 & -\ell_y \end{bmatrix} \tag{3.10}$$

The nonlinear parts are,

$$g(x) = \begin{bmatrix} 0 \\ a_3 r^3 - \frac{b}{3\delta_{sat}^2}\delta_0^3 \\ \frac{1}{6}\Psi^3 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{3.11}$$

If we introduce the transformation matrix (T) of eigenvectors of A evaluated at the bifurcation point,

$$T = [m_{ij}] \qquad i,j = 1,2,3,4,5,6 \tag{3.12}$$

$$T^{-1} = [n_{ij}] \qquad i,j = 1,2,3,4,5,6 \tag{3.13}$$

the linear change of coordinates,

$$x = Tz\,, \qquad z = T^{-1}x \tag{3.14}$$

transforms system (3.7) into its normal form

$$\dot{z} = T^{-1}ATz + T^{-1}g(Tz) \tag{3.15}$$

With this transformation we get,

$$T^{-1}AT = \begin{bmatrix} 0 & -w_0 & 0 & 0 & 0 & 0 \\ w_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & P_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & P_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & P_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & P_4 \end{bmatrix} \tag{3.16}$$

Using center manifold theory we get, as shown in [Chow and Mallet-Paret (1977)],

$$g = \begin{bmatrix} 0 \\ \ell_{21}z_1^3 + \ell_{22}z_2^3 + \ell_{23}z_1^2z_2 + \ell_{24}z_1z_2^2 \\ \ell_{31}z_1^3 + \ell_{32}z_2^3 + \ell_{33}z_1^2z_2 + \ell_{34}z_1z_2^2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{3.17}$$

Substitution of Equation (3.14) into Equation (3.7) yields,

$$\dot{z}_1 = -w_0z_2 + r_{11}z_1^3 + r_{12}z_1^2z_2 + r_{13}z_1z_2^2 + r_{14}z_2^3 \tag{3.18}$$

$$\dot{z}_2 = w_0z_1 + r_{21}z_1^3 + r_{22}z_1^2z_2 + r_{23}z_1z_2^2 + r_{24}z_2^3 \tag{3.19}$$

2. Integral Averaging

We write Equations (3.18) and (3.19) in the form

$$\dot{z}_1 = -w_0z_2 + F_1(z_1, z_2)\,, \tag{3.20}$$

$$\dot{z}_2 = w_0z_1 + F_2(z_1, z_2) \tag{3.21}$$

If we introduce polar coordinates in the form,

$$z_1 = R\cos\theta\,, \qquad z_2 = R\sin\theta \tag{3.22}$$

Equations (3.20), (3.21) result in

$$\dot{R} = F_1(R,\theta)\cos\theta + F_2(R,\theta)\sin\theta \tag{3.23}$$

$$R\dot{\theta} = w_0R + F_2(R,\theta)\cos\theta - F_1(R,\theta)\sin\theta \tag{3.24}$$

Equation (3.23) yields

$$\dot{R} = P(\theta)R^3 \tag{3.25}$$

where $P(\theta)$ is a 2π-periodic function in the angular coordinate θ. If Equation (3.25) is averaged over one cycle in θ, we get an equation with constant coefficients,

$$\dot{R} = KR^3 \tag{3.26}$$

where,

$$K = \frac{1}{2\pi}\int_0^{\pi} P(\theta) \cdot d\theta \tag{3.27}$$

Equation (3.27) is simplified after evaluation of the integral as,

$$K = \frac{1}{8}\left[3r_{11} + r_{13} + r_{22} + 3r_{24}\right] \tag{3.28}$$

where the coefficients are as follows,

$$\begin{aligned}
r_{11} &= n_{12}\ell_{21} + n_{13}\ell_{31} \\
r_{13} &= n_{12}\ell_{24} + n_{13}\ell_{34} \\
r_{22} &= n_{22}\ell_{23} + n_{23}\ell_{33} \\
r_{24} &= n_{22}\ell_{22} + n_{23}\ell_{32}
\end{aligned}$$

where the n_{12}, n_{13}, n_{22}, and n_{23} are the elements of the inverse of transformation matrix T. The values of the coefficients ℓ_{ij} are given by the following expressions

$$\begin{aligned}
\ell_{21} = {} & a_2 m_{21}^3 - b_\ell \Big[c^3 K_\psi^3 m_{41}^3 + K_r^3 m_{51}^3 + K_y^3 m_{61}^3 + 3c^2 K_\psi^2 K_r m_{41}^2 m_{51} \\
& + 3c^2 K_\psi^2 K_y m_{41}^2 m_{61} + 3cK_\psi K_r^2 m_{51}^2 m_{41} + 3cK_\psi K_y^2 m_{61}^2 m_{41} + 3K_r K_y^2 m_{61}^2 m_{51} \\
& + 3K_r^2 K_y m_{51}^2 m_{61} + 6cK_\psi K_r K_y m_{41} m_{51} m_{61}\Big]
\end{aligned}$$

$$
\begin{aligned}
\ell_{22} &= a_3 m_{22}^2 - b_\ell \Big[c^3 K_\psi^3 m_{42}^3 + K_r^3 m_{52}^3 + K_y^3 m_{62}^3 + 3c^2 K_\psi^2 K_r m_{42}^2 m_{52} \\
&\quad + 3c^2 K_\psi^2 K_y m_{42}^2 m_{62} + 3cK_\psi K_r^2 m_{52}^2 m_{42} + 3cK_\psi K_y^2 m_{62}^2 m_{42} + 3K_r K_y^2 m_{62}^2 m_{52} \\
&\quad + 3K_r^2 K_y m_{52}^2 m_{62} + 6cK_\psi K_r K_y m_{42} m_{52} m_{62} \Big] \\
\ell_{23} &= 3a_3 m_{21}^2 m_{22} - b_\ell \Big[3c^3 K_\psi^3 m_{41}^2 m_{42} + 3K_r^3 m_{51}^2 m_{52} + 3K_y^3 m_{61}^2 m_{62} \\
&\quad + 3c^2 K_\psi^2 K_r \left(m_{41}^2 m_{52} + 2m_{41} m_{42} m_{51} \right) + 3c^2 K_\psi^2 K_y \left(m_{41}^2 m_{62} + 2m_{41} m_{42} m_{61} \right) \\
&\quad + 3cK_\psi K_r^2 \left(m_{51}^2 m_{42} + 2m_{51} m_{52} m_{41} \right) + 3cK_\psi K_y^2 \left(m_{61}^2 m_{42} + 2m_{61} m_{62} m_{41} \right) \\
&\quad + 3K_r K_y^2 \left(m_{61}^2 m_{52} + 2m_{61} m_{62} m_{51} \right) + 3K_r^2 K_y \left(m_{51}^2 m_{62} + 2m_{51} m_{52} m_{61} \right) \\
&\quad + 6cK_\psi K_r K_y \left(m_{41} m_{51} m_{62} + \left(m_{41} m_{52} + m_{42} m_{51} \right) m_{61} \right) \Big] \\
\ell_{24} &= 3a_3 m_{21}^2 m_{22}^2 - b_\ell \Big[3c^3 K_\psi^3 m_{41} m_{42}^2 + 3K_r^3 m_{51} m_{52}^2 + 3K_y^3 m_{61} m_{62}^2 \\
&\quad + 3c^2 K_\psi^2 K_r \left(m_{42}^2 m_{51} + 2m_{41} m_{42} m_{52} \right) + 3c^2 K_\psi^2 K_y \left(m_{42}^2 m_{61} + 2m_{41} m_{42} m_{62} \right) \\
&\quad + 3cK_\psi K_r^2 \left(m_{52}^2 m_{41} + 2m_{51} m_{52} m_{42} \right) + 3cK_\psi K_y^2 \left(m_{62}^2 m_{41} + 2m_{61} m_{62} m_{42} \right) \\
&\quad + 3K_r K_y^2 \left(m_{62}^2 m_{51} + 2m_{61} m_{62} m_{52} \right) + 3K_r^2 K_y \left(m_{52}^2 m_{61} + 2m_{51} m_{52} m_{62} \right) \\
&\quad + 6cK_\psi K_r K_y \left(m_{42} m_{52} m_{61} + \left(m_{41} m_{52} + m_{42} m_{51} \right) m_{62} \right) \Big]
\end{aligned}
$$

$$
\begin{aligned}
\ell_{31} &= -\frac{1}{6} m_{11}^3 \\
\ell_{32} &= -\frac{1}{6} m_{12}^3 \\
\ell_{33} &= -\frac{1}{2} m_{11}^2 m_{12} \\
\ell_{34} &= -\frac{1}{2} m_{11} m_{12}^2 \\
b_\ell &= \frac{b}{3\delta_{sat}^2}
\end{aligned}
$$

C. RESULTS

The value of K is important for us to determine whether the bifurcation is supercritical ($K < 0$) or subcritical ($K > 0$). In this study we wanted to see the effects of observer dynamics to our system, especially the value of K. To do that we used the Fortran code (Appendix A) for the numerical results. The result was that the value of K was not affected by changes in the observer natural frequency. The reason for this can be traced back to the definition of K, Equation (3.28). It can be seen that in the definitions for r_{ij} and ℓ_{ij} only the first two eigenvectors m_{i1}, m_{i2} for $i = 1, 2, \ldots 6$ of the A matrix, Equation (3.10) appear. Therefore, we have to show that these remain the same for all observer natural frequencies.

This A matrix, Equation (3.10), actually consists of 4 block matrices, each 3×3, which are the same as shown in Equation (2.22). Let us denote the A matrix as follows,

$$A = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ 0 & \mathcal{C} \end{bmatrix}$$

The eigenvalues of A can be computed by

$$\begin{vmatrix} \mathcal{A} - \lambda I & \mathcal{B} \\ 0 & \mathcal{C} - \lambda I \end{vmatrix} = 0$$

or

$$|\mathcal{A} - \lambda I| \cdot |\mathcal{C} - \lambda I| = 0$$

We can group the eigenvalues in two different groups: $\lambda_{1,i}$ for $i = 1, 2, 3$ are the eigenvalues of $\mathcal{A}$ an $\lambda_{2,i}$ for $i = 1, 2, 3$ are the eigenvalues of $\mathcal{C}$. The eigenvectors associated with these eigenvalues can be found as follows.

For $\lambda = \lambda_{1,i}$,

$$\begin{bmatrix} \mathcal{A} - \lambda_{1,i} I & \mathcal{B} \\ 0 & \mathcal{C} - \lambda_{1,i} I \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and

$$
\begin{aligned}
[\mathcal{A} - \lambda_{1,i} \mathcal{I}][v_1] + [\mathcal{B}][v_2] &= 0 \\
[0][v_1] + [\mathcal{C} - \lambda_{1,i} \mathcal{I}][v_2] &= 0
\end{aligned}
$$

Since $\lambda_{1,i}$ is an eigenvalue of $\mathcal{A}$ and the eigenvalues of $\mathcal{A}$ and $\mathcal{C}$ are distinct, the matrix $[\mathcal{C} - \lambda_{1,i} \mathcal{I}]$ is nonsingular which means that $[v_2] = 0$. Therefore, we get

$$[\mathcal{A} - \lambda_{1,i} \mathcal{I}][v_1] = 0$$

which means that v_1 is an eigenvector of $\mathcal{A}$. Therefore, the first three eigenvectors of A are the same as the eigenvectors of $\mathcal{A}$ and they are independent of the dynamics of the observer. Of course, the remaining three eigenvectors of A depend on the oberver natural frequency, but, as we pointed out earlier, none of these appear in the definition of the nonlinear stability coefficient K.

IV. COMPENSATOR IN A DIFFERENT BASIS

A. CRITICAL VALUE OF C

If we look at Equation (3.6), we can see that the basis for our system was $[X, \tilde{X}]$. Now we are going to represent our system in $[X, \hat{X}]$ basis where $\hat{X}$ is the estimate of X. In this compensator basis the critical value of C in Equation (4.3) is no longer constant. Therefore, we used a Fortran code (Appendix B) to calculate the critical C values for different observer natural frequencies. A plot of these critical C values for different observer natural frequencies can be seen in Fig. 4.1. The observer natural frequencies are in nondimensional seconds whereas the control natural frequencies are normalized with respect to the corresponding observer natural frequencies. The system is unstable for values of C less than the critical value.

B. THIRD ORDER EXPANSIONS OF THE SYSTEM EQUATIONS

1. Perturbation in K_ϕ

In the previous chapters we worked on the $[X, \tilde{X}]$ basis. Now we are going to represent our system in the new basis, which is $[X, \hat{X}]$, where $\hat{X}$ is the estimate of X. Our equations of motion were,

$$\begin{aligned} \dot{\Psi} &= r \\ \dot{r} &= ar + a_3 r^3 + b\delta \\ \dot{y} &= \sin\Psi \end{aligned} \tag{4.1}$$

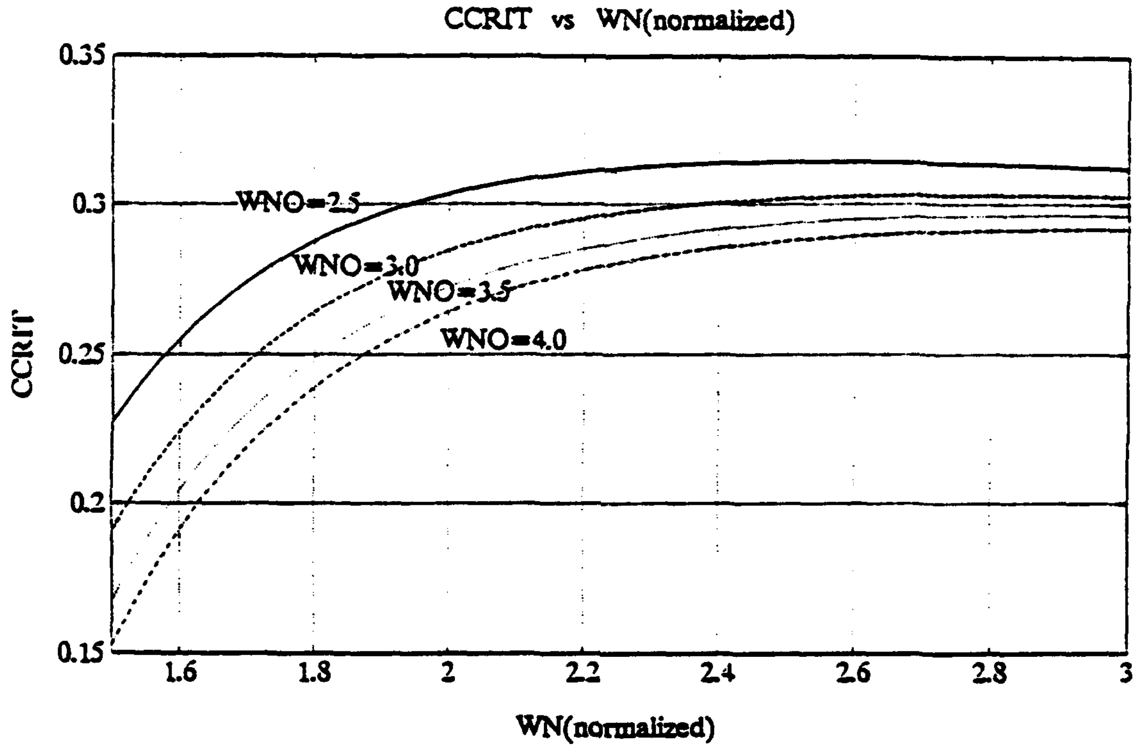

Figure 4.1: C_{crit} versus natural frequency for K_ψ.

where

$$\delta = \delta_{sat} \tanh\left(\frac{\delta_0}{\delta_{sat}}\right) \tag{4.2}$$

$$\delta_0 = CK_\psi \hat{\Psi} + K_r \hat{r} + K_y \hat{y} \tag{4.3}$$

or in compact form,

$$\dot{X} = f(x)\,, \qquad X = \left[\Psi, r, y, \hat{\Psi}, \hat{r}, \hat{y}\right]^T \tag{4.4}$$

This system can be written in the form

$$\dot{X} = AX + g(x) \tag{4.5}$$

A is the Jacobian matrix of $f(x)$ evaluated at $X = 0$, and $g(x)$ contains all non-linear terms of Equation (4.1). Taylor expansion of the nonlinear terms about the equilibrium, where we keep the first non-vanishing nonlinear coefficients only, gives

$$\sin\Psi = \Psi - \frac{1}{6}\Psi^3 \tag{4.6}$$

$$\delta = \delta_0 - \frac{1}{3\delta_{sat}^2}\delta_0^3 \tag{4.7}$$

Substitution of Equations (4.6) and (4.7) into Equation (4.1) gives us the A matrix in Equation (4.5) as follows

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & bcK_\Psi & bK_r & bK_y \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \ell_\Psi & 0 & 1 & -\ell_\Psi \\ 0 & 0 & \ell_r & bcK_\Psi & bK_r & -\ell_r + bK_y \\ 0 & 0 & \ell_y & 1 & 0 & -\ell_y \end{bmatrix} \tag{4.8}$$

The nonlinear parts are,

$$g(x) = \begin{bmatrix} 0 \\ a_3 r^3 - \frac{b}{3\delta_{sat}^2}\delta_0^3 \\ -\frac{1}{6}\Psi^3 \\ 0 \\ -\frac{b}{3\delta_{sat}^2}\delta_0^3 \\ 0 \end{bmatrix} \tag{4.9}$$

If we introduce the transformation matrix (T) of eigenvectors of A evaluated at the bifurcation point,

$$T = [m_{ij}] \qquad i,j = 1,2,3,4,5,6 \tag{4.10}$$

$$T^{-1} = [n_{ij}] \qquad i,j = 1,2,3,4,5,6 \tag{4.11}$$

the linear change of coordinates,

$$x = Tz\ , \qquad z = T^{-1}x \tag{4.12}$$

transforms system (4.5) into its normal form

$$\dot{z} = T^{-1}ATz + T^{-1}g(Tz) \tag{4.13}$$

At the bifurcation point

$$T^{-1}AT = \begin{bmatrix} 0 & -w_0 & 0 & 0 & 0 & 0 \\ w_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & P_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & P_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & P_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & P_4 \end{bmatrix} \tag{4.14}$$

with $w_0 > 0$ and $P_i < 0$.

Using center manifold theory we get, as shown in [Chow and Mallet-Paret (1977)],

$$g = \begin{bmatrix} 0 \\ \ell_{21}z_1^3 + \ell_{22}z_2^3 + \ell_{23}z_1^2 z_2 + \ell_{24}z_1 z_2^2 \\ \ell_{31}z_1^3 + \ell_{32}z_2^3 + \ell_{33}z_1^2 z_2 + \ell_{34}z_1 z_2^2 \\ 0 \\ \ell_{51}z_1^3 + \ell_{52}z_2^3 + \ell_{53}z_1^2 z_2 + \ell_{54}z_1 z_2^2 \\ 0 \end{bmatrix} \tag{4.15}$$

Substitution of Equation (4.12) into Equation (4.5) yields,

$$\dot{z}_1 = -w_0 z_2 + r_{11}z_1^3 + r_{12}z_1^2 z_2 + r_{13}z_1 z_2^2 + r_{14}z_2^3 \tag{4.16}$$

$$\dot{z}_2 = w_0 z_1 + r_{21}z_1^3 + r_{22}z_1^2 z_2 + r_{23}z_1 z_2^2 + r_{24}z_2^3 \tag{4.17}$$

where the terms r_{ij} are evaluated by a Fortran code (Appendix C).

For values of C close to its critical value, Equations (4.16) and (4.17) become,

$$\dot{z}_1 = \alpha'\varepsilon z_1 - (w_0 + w'\varepsilon)z_2 + r_{11}z_1^3 + r_{12}z_1^2 z_2 + r_{13}z_1 z_2^2 + r_{14}z_2^3 \tag{4.18}$$

$$\dot{z}_2 = (w_0 + w'\varepsilon)z_1 + \alpha'\varepsilon z_2 + r_{21}z_1^3 + r_{22}z_1^2 z_2 + r_{23}z_1 z_2^2 + r_{24}z_2^3 \tag{4.19}$$

where ε is the difference between C and its critical value C_*. The terms α' and w' denote the derivative of the real and imaginary part, respectively, of the critical pair of the eigenvalues with respect to C evaluated at C_*

2. Integral Averaging

We write Equations (4.18) and (4.19) in the form,

$$\dot{z}_1 = \alpha'\varepsilon z_1 - (w_0 + w'\varepsilon)z_2 + F_1(z_1, z_2) \tag{4.20}$$

$$\dot{z}_2 = (w_0 + w'\varepsilon)z_1 + \alpha'\varepsilon z_2 + F_2(z_1, z_2) \tag{4.21}$$

If we introduce polar coordinates in the form,

$$z_1 = R\cos\theta\ , \qquad z_2 = R\sin\theta \tag{4.22}$$

Equations (4.20), (4.21) result in

$$\dot{R} = \alpha'\varepsilon R + F_1(R,\theta)\cos\theta + F_2(R,\theta)\sin\theta \tag{4.23}$$

$$R\dot{\theta} = (w_0 + w'\varepsilon)R + F_2(R,\theta)\cos\theta - F_1(R,\theta)\sin\theta \tag{4.24}$$

Equation (4.23) yields

$$\dot{R} = \alpha'\varepsilon R + P(\theta)R^3 \tag{4.25}$$

where $P(\theta)$ is a 2π-periodic function in the angular coordinate θ. If Equation (4.25) is averaged over one cycle in θ [Chow and Mallet-Paret (1977)], we get an equation with constant coefficients,

$$\dot{R} = \alpha'\varepsilon R + KR^3 \tag{4.26}$$

where

$$K = \frac{1}{2\pi}\int_0^{\pi} P(\theta)d\theta \tag{4.27}$$

Equation (4.27) is simplified after evaluation of the integral as,

$$K = \frac{1}{8}\left[3r_{11} + r_{13} + r_{22} + 3r_{24}\right] \tag{4.28}$$

where the coefficients are as follows

$$
\begin{aligned}
r_{11} &= n_{12}\ell_{21} + n_{13}\ell_{31} + n_{15}\ell_{51} \\
r_{13} &= n_{12}\ell_{24} + n_{13}\ell_{34} + n_{15}\ell_{54} \\
r_{22} &= n_{22}\ell_{23} + n_{23}\ell_{33} + n_{25}\ell_{53} \\
r_{24} &= n_{22}\ell_{22} + n_{23}\ell_{32} + n_{25}\ell_{52}
\end{aligned}
$$

where the n_{12}, n_{13}, n_{15}, n_{22}, n_{23}, and n_{25} are the elements of the inverse of transformation matrix T. The values of the coefficients ℓ_{21}, ℓ_{22}, ℓ_{23}, ℓ_{24}, ℓ_{31}, ℓ_{32}, ℓ_{33}, and ℓ_{34} are the same as in Chapter III. The values of the other ℓ_{ij} coefficients are given by the following expressions:

$$
\begin{aligned}
\ell_{51} = & -b_\ell \Big[c^3K_\phi^3 m_{41}^3 + K_r^3 m_{51}^3 + K_y^3 m_{61}^3 + 3c^2K_\phi^2 K_r m_{41}^2 m_{51} \\
& +3c^2K_\phi^2 K_y m_{41}^2 m_{61} + 3cK_\phi K_r^2 m_{51}^2 m_{41} + 3cK_\phi K_y^2 m_{61}^2 m_{41} + 3K_r K_y^2 m_{61}^2 m_{51} \\
& +3K_r^2 K_y m_{51}^2 m_{61} + 6cK_\phi K_r K_y m_{41} m_{51} m_{61}\Big] \\
\ell_{52} = & -b_\ell \Big[c^3K_\phi^3 m_{42}^3 + K_r^3 m_{52}^3 + K_y^3 m_{62}^3 + 3c^2K_\phi^2 K_r m_{42}^2 m_{52} \\
& +3c^2K_\phi^2 K_y m_{42}^2 m_{62} + 3cK_\phi K_r^2 m_{52}^2 m_{42} + 3cK_\phi K_y^2 m_{62}^2 m_{42} + 3K_r K_y^2 m_{62}^2 m_{52} \\
& +3K_r^2 K_y m_{52}^2 m_{62} + 6cK_\phi K_r K_y m_{42} m_{52} m_{62}\Big] \\
\ell_{53} = & -b_\ell \Big[3c^3K_\phi^3 m_{41}^2 m_{42} + 3K_r^3 m_{51}^2 m_{52} + 3K_y^3 m_{61}^2 m_{62} \\
& +3c^2K_\phi^2 K_r \left(m_{41}^2 m_{52} + 2m_{41}m_{42}m_{51}\right) + 3c^2K_\phi^2 K_y \left(m_{41}^2 m_{62} + 2m_{41}m_{42}m_{61}\right) \\
& +3cK_\phi K_r^2 \left(m_{51}^2 m_{42} + 2m_{51}m_{52}m_{41}\right) + 3cK_\phi K_y^2 \left(m_{61}^2 m_{42} + 2m_{61}m_{62}m_{41}\right) \\
& +3K_r K_y^2 \left(m_{61}^2 m_{52} + 2m_{61}m_{62}m_{51}\right) + 3K_r^2 K_y \left(m_{51}^2 m_{62} + 2m_{51}m_{52}m_{61}\right) \\
& +6cK_\phi K_r K_y \left(m_{41}m_{51}m_{62} + \left(m_{41}m_{52} + m_{42}m_{51}\right) m_{61}\right)\Big]
\end{aligned}
$$

$$
\begin{aligned}
\ell_{54} = & -b_\ell \Big[3c^3K_\phi^3 m_{41}m_{42}^2 + 3K_r^3 m_{51}m_{52}^2 + 3K_y^3 m_{61}m_{62}^2 \\
& +3c^2K_\phi^2K_r\left(m_{42}^2m_{51} + 2m_{41}m_{42}m_{52}\right) + 3c^2K_\phi^2K_y\left(m_{42}^2m_{61} + 2m_{41}m_{42}m_{62}\right) \\
& +3cK_\phi K_r^2\left(m_{52}^2m_{41} + 2m_{51}m_{52}m_{42}\right) + 3cK_\phi K_y^2\left(m_{62}^2m_{41} + 2m_{61}m_{62}m_{42}\right) \\
& +3K_rK_y^2\left(m_{62}^2m_{51} + 2m_{61}m_{62}m_{52}\right) + 3K_r^2K_y\left(m_{52}^2m_{61} + 2m_{51}m_{52}m_{62}\right) \\
& +6cK_\phi K_rK_y\left(m_{42}m_{52}m_{61} + \left(m_{41}m_{52} + m_{42}m_{51}\right)m_{62}\right)\Big]
\end{aligned}
$$

C. RESULTS

Existence and stability of limit cycles can be determined by analyzing the equilibrium points of the averaged Equation (4.26), which correspond to periodic solutions in z_1, z_2 as can be seen from Equation (4.22). From Equation (4.26) we can easily see that:

1. If $\alpha' > 0$, then
   (a) if $K > 0$, then unstable periodic solutions co-exist with the stable equilibrium for $\varepsilon < 0$, and
   (b) if $K < 0$, then stable periodic solutions co-exist with the unstable equilibrium for $\varepsilon > 0$.
2. If $\alpha' < 0$, then
   (a) if $K > 0$, then unstable periodic solutions co-exist with the stable equilibrium for $\varepsilon > 0$, and
   (b) if $K < 0$, then stable periodic solutions co-exist with the unstable equilibrium for $\varepsilon < 0$.

We refer to $K < 0$ as the supercritical, and $K > 0$ as the subcritical PAH bifurcation. In the supercritical case, after the equilibrium state loses its stability the system converges to a stable periodic solution with amplitude which increases continuously as the difference ε is increased.

In the subcritical case, however, before the equilibrium state loses stability, its domain of attraction becomes very small since it is bounded by the amplitudes of the unstable limit cycles. In such a case, an initial disturbance of sufficient magnitude can throw the system off the nominal path even before its domain of attraction has completely shrunk to zero. As the nominal equilibrium becomes unstable, the system jumps to a different state of motion with a locally, at $\varepsilon = 0$, discontinuous increase in the amplitude [Papoulias (1993)].

In our case, the value of α' is always negative, which can be seen easily from Figure 4.1. If we look at the nature of the curve for the critical value of C for different natural frequencies, we will see that as the value of critical C decreases for the same natural frequency, the system becomes unstable.

After using a Fortran code (Appendix C) we observed that the nonlinear stability coefficient K depends on observer dynamics. Figure 4.2 shows that for a given control design, the observer must be as responsive as possible to ensure negative K (stable limit cycle). On the other hand, for a given observer design, if the control law is too slow we get subcritical behavior (unstable limit cycle).

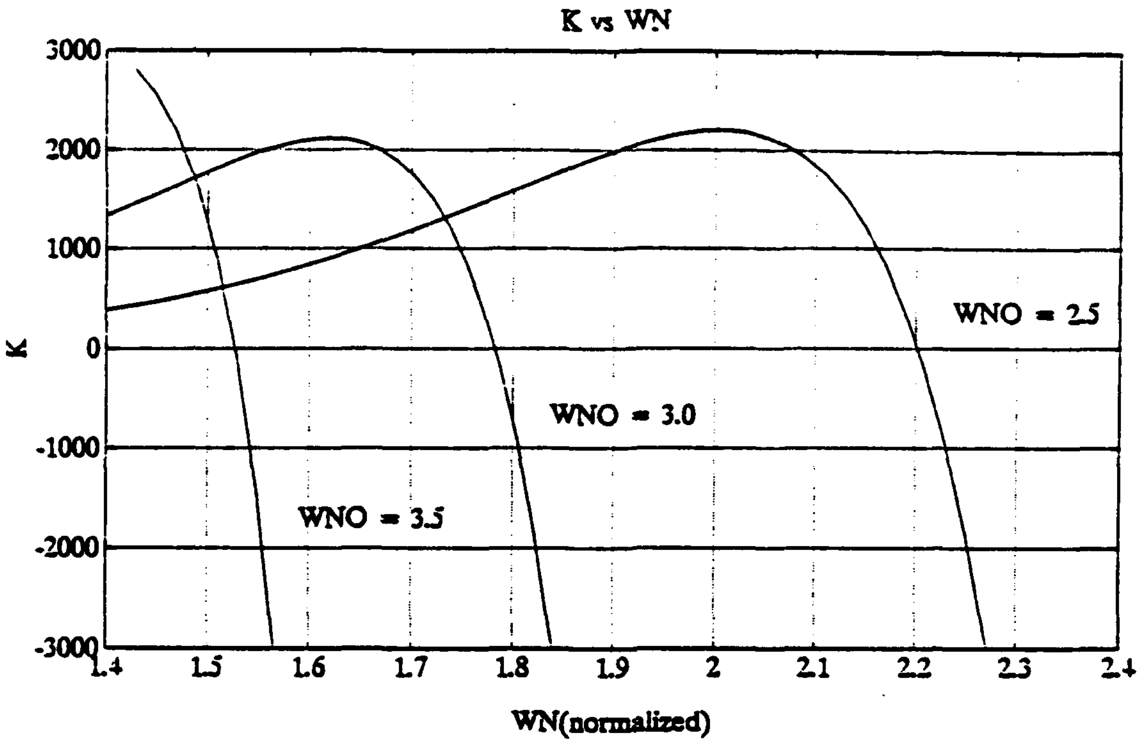

Figure 4.2: K_{K_ψ} versus w_n for different observer w_n.

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V. CONCLUSIONS AND RECOMMENDATIONS

A. CONCLUSIONS

An investigation of the nonlinear dynamic response characteristics of a marine vehicle has been presented. Particular emphasis in this work was placed on analyzing the effects of observer design on system response after initial loss of stability of straight line motion. Bifurcation theory techniques were utilized in order to assess that behavior. The main conclusions of this work can be summarized as follows.

1. There exists a critical point for a certain combination of system gains and system parameters for stability of straight line motion. The loss of stability occurs generically in the form of Poincare-Andronov-Hopf bifurcations. As the parameter crosses its critical value, a family of periodic orbits, self sustained oscillations develops. Center manifold reduction and integral averaging techniques were used in order to establish the direction of the bifurcation and stability of the resulting periodic solutions [Papoulias, Oral (1993)].

2. For $[X, \tilde{X}]$ basis the critical point does not depend on the observer dynamics (separation principle). The nonlinear stability coefficient K was not influenced by observer dynamics. The previous reduction process shows that K depends on the first two eigenvectors of the 6×6 matrix A. Matrix algebra shows that these eigenvectors are associated only with the controller dynamics.

3. For $[X, \hat{X}]$ basis the critical value depends on observer dynamics. For a given control design, the observer must be as responsive as possible to maximize the

region of stability. On the other hand, for a given observer design, the control must be as slow as possible to maximize region of stability.

4. The nonlinear stability coefficient K depends on observer dynamics for this basis. For a given control design, the observer must be as responsive as possible to ensure negative K (stable limit cycles). In this benign loss of stability the resulting periodic solutions are continuous single-valued functions of the parameter distance from its critical value. On the other hand, for a given observer design, if the control law is too slow we get subcritical behavior (unstable limit cycles). In such a case, the periodic solutions develop with what appears to be a discontinuous increase in the amplitude of oscillations [Papoulias, Oral (1993)].

B. RECOMMENDATIONS

The differences between the two bases with respect to robustness properties of the system have to be analyzed.

APPENDIX A

HOPF BIFURCATION PROGRAM FOR $[X, \tilde{X}]$ BASIS

```
      PROGRAM HTKPSI
C     HOPF BIFURCATIONS
C     NOMOTO'S FIRST ORDER MODEL
C
C     CALCULATIONS FOR K AND CCRITICAL IF KPSI CHANGES W/ C
C
C234567
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
C
      DOUBLE PRECISION K1,K2,K,LPSI,LY,LR,IZ,L,
     &              MASS,NV,NR,NVDOT,NRDOT,NDRS,NDRB,KPSI,KR,KY,K3,
     &             L21,L22,L23,L24,L31,L32,L33,L34,L51,L52,L53,L54,
     &             M11,M12,M13,M14,M15,M16,M21,M22,M23,M24,M25,M26,
     &             M31,M32,M33,M34,M35,M36,M41,M42,M43,M44,M45,M46,
     &             M51,M52,M53,M54,M55,M56,M61,M62,M63,M64,M65,M66,
     &             N11,N12,N13,N14,N15,N16,N21,N22,N23,N24,N25,N26,
     &             N31,N32,N33,N34,N35,N36,N41,N42,N43,N44,N45,N46,
     &              N51,N52,N53,N54,N55,N56,N61,N62,N63,N64,N65,N66
C
      DIMENSION AMAT(6,6),T(6,6),TINV(6,6),FV1(6),IV1(6),YYY(6,6)
      DIMENSION WR(6),WI(6),TSAVE(6,6),TLUD(6,6),IVLUD(6),SVLUD(6)
      DIMENSION ASAVE(6,6),A1(6,6),A2(6,6)
      OPEN (11,FILE='AKPSI.MAT',STATUS='NEW')
C
      WEIGHT=435.0
      IZ    =45.0
      L     =7.3
      RHO   =1.94
      G     =32.2
      XG    =0.0104
      MASS  =WEIGHT/G
      MASS  =MASS/(0.5*RHO*L**3)
      IZ    =IZ/(0.5*RHO*L**5)
      XG    =XG/L
      YRDOT =-0.00000
      YVDOT =-0.03430
      YR    =+0.00000
      YV    =-0.10700
      YDRS  =+0.01241
      YDRB  =+0.01241
      NRDOT =-0.00047
      NVDOT =-0.00000
      NR    =-0.00390
```

```
      NV     =-0.00000
      NDRS   =-0.337*YDRS
      NDRB   =+0.283*YDRS
      DH   =(IZ-NRDOT)*(MASS-YVDOT)-
     &      (MASS*XG-YRDOT)*(MASS*XG-NVDOT)
      A11=((IZ-NRDOT)*YV-(MASS*XG-YRDOT)*NV)/DH
      A12=((IZ-NRDOT)*(-MASS+YR)-
     &      (MASS*XG-YRDOT)*(-MASS*XG+NR))/DH
      A21=((MASS-YVDOT)*NV-(MASS*XG-NVDOT)*YV)/DH
      A22=((MASS-YVDOT)*(-MASS*XG+NR)-
     &      (MASS*XG-NVDOT)*(-MASS+YR))/DH
      B11=((IZ-NRDOT)*YDRS-(MASS*XG-YRDOT)*NDRS)/DH
      B12=((IZ-NRDOT)*YDRB-(MASS*XG-YRDOT)*NDRB)/DH
      B21=((MASS-YVDOT)*NDRS-(MASS*XG-NVDOT)*YDRS)/DH
      B22=((MASS-YVDOT)*NDRB-(MASS*XG-NVDOT)*YDRB)/DH
      B1 =B11-B12
      B2 =B21-B22
C
  200 WRITE (*,1004)
      READ  (*,*)    WNMIN,WNMAX,IWN
      INCR=IWN
      WRITE (*,*) 'ENTER OBSERVER WN'
      READ (*,*) WNO
  205 WRITE(*,1007)
      READ (*,*) A3
C     D0 is Dsat

   50 WRITE (*,1006)
      READ  (*,*)    D0
      WRITE (*,1008)
      READ  (*,*)  CCRIT
      C1=(A11*A22-A21*A12)*(A21*B1-A11*B2)
      C2=(A11+A22)*(A21*B1-A11*B2)+B2*(A11*A22-A21*A12)
      C3=-(A21*B1-A11*B2)**2
      A=C1/C2
      B=C3/C2
C
      DO 1 II=1,INCR
C
C
      WN   =WNMIN+(WNMAX-WNMIN)*(II-1)/(INCR-1)
C      print *,wn
      ALPHA0=WN**3
      ALPHA1=2.15*WN**2
      ALPHA2=1.75*WN
      KPSI=-ALPHA1/B
      KY   =-ALPHA0/B
      KR   =-(ALPHA2+A)/B
      GAMA0=WNO**3
      GAMA1=2.15*WNO**2
```

```
      GAMA2=1.75*WNO
      LY=A+GAMA2
      LPSI=A*LY+GAMA1
      LR=A*LPSI+GAMA0
C234567
C    A    MATRIX
      AMAT(1,1)=0.0
      AMAT(1,2)=1.0
      AMAT(1,3)=0.0
      AMAT(1,4)=0.0
      AMAT(1,5)=0.0
      AMAT(1,6)=0.0
      AMAT(2,1)=B*CCRIT*KPSI
      AMAT(2,2)=A+(B*KR)
      AMAT(2,3)=B*KY
      AMAT(2,4)=-B*CCRIT*KPSI
      AMAT(2,5)=-B*KR
      AMAT(2,6)=-B*KY
      AMAT(3,1)=1.0
      AMAT(3,2)=0.0
      AMAT(3,3)=0.0
      AMAT(3,4)=0.0
      AMAT(3,5)=0.0
      AMAT(3,6)=0.0
      AMAT(4,1)=0.0
      AMAT(4,2)=0.0
      AMAT(4,3)=0.0
      AMAT(4,4)=0.0
      AMAT(4,5)=1.0
      AMAT(4,6)=-LPSI
      AMAT(5,1)=0.0
      AMAT(5,2)=0.0
      AMAT(5,3)=0.0
      AMAT(5,4)=0.0
      AMAT(5,5)=A
      AMAT(5,6)=-LR
      AMAT(6,1)=0.0
      AMAT(6,2)=0.0
      AMAT(6,3)=0.0
      AMAT(6,4)=1.0
      AMAT(6,5)=0.0
      AMAT(6,6)=-LY
      DO 11 I=1,6
        DO 12 J=1,6
          ASAVE(I,J)=AMAT(I,J)
12      CONTINUE
11    CONTINUE
      CALL RG(6,6,AMAT,WR,WI,1,YYY,IV1,FV1,IERR)
      CALL DSOMEG(IEV,WR,WI,OMEGA,CHECK)
C      WRITE (*,*) IEV
```

```
      WRITE (60,*) (WR(IREAL),IREAL=1,6)
      OMEGA0=OMEGA
      DO 5 I=1,6
        T(I,1)=YYY(I,IEV)
        T(I,2)=-YYY(I,IEV+1)
5     CONTINUE
      IF(IEV.EQ.1) GO TO 13
      IF(IEV.EQ.2) GO TO 14
      IF(IEV.EQ.3) GO TO 15
      IF(IEV.EQ.4) GO TO 16
      IF(IEV.EQ.5) GO TO 17
      STOP 3004
13    DO 21 I=1,6
        T(I,3)=YYY(I,3)
        T(I,4)=YYY(I,4)
        T(I,5)=YYY(I,5)
        T(I,6)=YYY(I,6)
21    CONTINUE
      GO TO 30
14    DO 22 I=1,6
        T(I,3)=YYY(I,1)
        T(I,4)=YYY(I,4)
        T(I,5)=YYY(I,5)
        T(I,6)=YYY(I,6)
22    CONTINUE
      GO TO 30
15    DO 23 I=1,6
        T(I,3)=YYY(I,1)
        T(I,4)=YYY(I,2)
        T(I,5)=YYY(I,5)
        T(I,6)=YYY(I,6)
23    CONTINUE
      GO TO 30
16    DO 24 I=1,6
        T(I,3)=YYY(I,1)
        T(I,4)=YYY(I,2)
        T(I,5)=YYY(I,3)
        T(I,6)=YYY(I,6)
24    CONTINUE
      GO TO 30
17    DO 25 I=1,6
        T(I,3)=YYY(I,1)
        T(I,4)=YYY(I,2)
        T(I,5)=YYY(I,3)
        T(I,6)=YYY(I,4)
25    CONTINUE
30    CONTINUE
C
C     NORMALIZATION OF THE CRITICAL EIGENVECTOR
C
```

```
      CALL NORMAL(T)
C
C     INVERT TRANSFORMATION MATRIX
C
      DO 2 I=1,6
        DO 3 J=1,6
          TINV(I,J)=0.0
          TSAVE(I,J)=T(I,J)
3       CONTINUE
2     CONTINUE
      CALL DLUD(6,6,TSAVE,6,TLUD,IVLUD)
      DO 4 I=1,6
        IF (IVLUD(I).EQ.0) STOP 3003
4     CONTINUE
      CALL DILU(6,6,TLUD,IVLUD,SVLUD)
      DO 8 I=1,6
        DO 9 J=1,6
          TINV(I,J)=TLUD(I,J)
9     CONTINUE
8     CONTINUE
C
C     CHECK  Inv(T)*A*T
C
      IMULT=1
      IF (IMULT.EQ.1) CALL MULT(TINV,ASAVE,T,A2)
      IF (IMULT.EQ.0) STOP 3007
      P1=A2(1,1)
      P2=A2(2,2)
      P=A2(3,3)
      Q=A2(4,4)
      R=A2(5,5)
      S=A2(6,6)
      WRITE(21,*)P1,P2,P,Q,R,S
C
C     DEFINITION OF Nij
C
      N11=TINV(1,1)
      N21=TINV(2,1)
      N31=TINV(3,1)
      N41=TINV(4,1)
      N51=TINV(5,1)
      N61=TINV(6,1)
      N12=TINV(1,2)
      N22=TINV(2,2)
      N32=TINV(3,2)
      N42=TINV(4,2)
      N52=TINV(5,2)
      N62=TINV(6,2)
      N13=TINV(1,3)
      N23=TINV(2,3)
```

```
      N33=TINV(3,3)
      N43=TINV(4,3)
      N53=TINV(5,3)
      N63=TINV(6,3)
      N14=TINV(1,4)
      N24=TINV(2,4)
      N34=TINV(3,4)
      N44=TINV(4,4)
      N54=TINV(5,4)
      N64=TINV(6,4)
      N15=TINV(1,5)
      N25=TINV(2,5)
      N35=TINV(3,5)
      N45=TINV(4,5)
      N55=TINV(5,5)
      N65=TINV(6,5)
      N16=TINV(1,6)
      N26=TINV(2,6)
      N36=TINV(3,6)
      N46=TINV(4,6)
      N56=TINV(5,6)
      N66=TINV(6,6)
C
C     DEFINITION OF Mij
C
      M11=T(1,1)
      M21=T(2,1)
      M31=T(3,1)
      M41=T(4,1)
      M51=T(5,1)
      M61=T(6,1)
      M12=T(1,2)
      M22=T(2,2)
      M32=T(3,2)
      M42=T(4,2)
      M52=T(5,2)
      M62=T(6,2)
      M13=T(1,3)
      M23=T(2,3)
      M33=T(3,3)
      M43=T(4,3)
      M53=T(5,3)
      M63=T(6,3)
      M14=T(1,4)
      M24=T(2,4)
      M34=T(3,4)
      M44=T(4,4)
      M54=T(5,4)
      M64=T(6,4)
      M15=T(1,5)
```

```
      M25=T(2,5)
      M35=T(3,5)
      M45=T(4,5)
      M55=T(5,5)
      M65=T(6,5)
      M16=T(1,6)
      M26=T(2,6)
      M36=T(3,6)
      M46=T(4,6)
      M56=T(5,6)
      M66=T(6,6)
      WRITE(70,*) N11,N12,N13
      WRITE(71,*) N14,N15,N16
      WRITE(72,*) N21,N22,N23
      WRITE(73,*) N24,N25,N26
      WRITE(74,*) N31,N32,N33
      WRITE(75,*) N34,N35,N36
      WRITE(76,*) N41,N42,N43
      WRITE(77,*) N44,N45,N46
      WRITE(78,*) N51,N52,N53
      WRITE(79,*) N54,N55,N56
      WRITE(80,*) N61,N62,N63
      WRITE(81,*) N64,N65,N66
C      K1=1./8.*((ALPHA2**3)+ALPHA0)/(ALPHA2)
C      K2=3.*A3-.5*(ALPHA2**2)/ALPHA0
C      K3=1./((B**2)*(D0**2))*(ALPHA2+A)*((ALPHA0/ALPHA2)+(A**2))
C      K=K1*(K2+K3)
C
C      print *, wn,k,ccrit
C
      BL=B/(3*D0**2)
C
C23456789012345678901234567890123456789012345678901234567890123456789012
      L21=A3*M21**3-BL*(CCRIT**3*KPSI**3*M41**3+KR**3*M51**3+
     &       KY**3*M61**3+
     &       3*CCRIT**2*KPSI**2*KR*M41**2*M51+
     &       3*CCRIT**2*KPSI**2*KY*M41**2*M61+
     & 3*CCRIT*KPSI*KR**2*M51**2*M41+3*CCRIT*KPSI*KY**2*M61**2*M41+
     &    3*KR*KY**2*M61**2*M51+3*KR**2*KY*M51**2*M61+
     &    6*CCRIT*KPSI*KR*KY*M41*M51*M61)
      L22=A3*M22**3-BL*(CCRIT**3*KPSI**3*M42**3+KR**3*M52**3+
     &       KY**3*M62**3+
     &       3*CCRIT**2*KPSI**2*KR*M42**2*M52+
     &       3*CCRIT**2*KPSI**2*KY*M42**2*M62+
     & 3*CCRIT*KPSI*KR**2*M52**2*M42+3*CCRIT*KPSI*KY**2*M62**2*M42+
     &    3*KR*KY**2*M62**2*M52+3*KR**2*KY*M52**2*M62+
     &    6*CCRIT*KPSI*KR*KY*M42*M52*M62)
      L23=3*A3*M21**2*M22-BL*(3*CCRIT**3*KPSI**3*M41**2*M42+
     &       3*KR**3*M51**2*M52+
```

```
     &        3*KY**3*M61**2*M62+
     &        3*CCRIT**2*KPSI**2*KR*(M41**2*M52+2*M41*M42*M51)+
     &      3*CCRIT**2*KPSI**2*KY*(M41**2*M62+2*M41*M42*M61)+
     &      3*CCRIT*KPSI*KR**2*(M51**2*M42+2*M51*M52*M41)+
     &      3*CCRIT*KPSI*KY**2*(M61**2*M42+2*M61*M62*M41)+
     &      3*KR*KY**2*(M61**2*M52+2*M61*M62*M51)+
     &      3*KR**2*KY*(M51**2*M62+2*M51*M52*M61)+
     &     6*CCRIT*KPSI**KR*KY*(M41*M51*M62+(M41*M52+M42*M51)*M61))
      L24=3*A3*M21*M22**2-BL*(3*CCRIT**3*KPSI**3*M41*M42**2+
     &        3*KR**3*M51*M52**2+
     &        3*KY**3*M61*M62**2+
     &        3*CCRIT**2*KPSI**2*KR*(M42**2*M51+2*M41*M42*M52)+
     &      3*CCRIT**2*KPSI**2*KY*(M42**2*M61+2*M41*M42*M62)+
     &      3*CCRIT*KPSI*KR**2*(M52**2*M41+2*M51*M52*M42)+
     &        3*CCRIT*KPSI*KY**2*(M62**2*M41+2*M61*M62*M42)+
     &      3*KR*KY**2*(M62**2*M51+2*M61*M62*M52)+
     &      3*KR**2*KY*(M52**2*M61+2*M51*M52*M62)+
     &      6*CCRIT*KPSI*KR*KY*(M42*M52*M61+(M41*M52+M42*M51)*M62))
      L31=(-1./6.)*M11**3
      L32=(-1./6.)*M12**3
      L33=(-1./2.)*M11**2*M12
      L34=(-1./2.)*M11*M12**2
      R11=N12*L21+N13*L31
      R12=N12*L23+N13*L33
      R13=N12*L24+N13*L34
      R14=N12*L22+N13*L32
      R21=N22*L21+N23*L31
      R22=N22*L23+N23*L33
      R23=N22*L24+N23*L34
      R24=N22*L22+N23*L32
C
      K=(3*R11+R13+R22+3*R24)/8
        WRITE (11,2001) WN,K,CCRIT
    1 CONTINUE
      STOP
 1004 FORMAT (' ENTER MIN,MAX, AND INCREMENTS OF
                 WN(stepstodivide)')
 1006 FORMAT (' ENTER DELTASAT.')
 1007 FORMAT (' ENTER A3')
 1008 FORMAT (' ENTER CCRIT')
 2001 FORMAT (3E15.5)
      END
C=================================================================
      SUBROUTINE DSOMEG(IJK,WR,WI,OMEGA,CHECK)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION WR(6),WI(6)
      CHECK=-1.0D+25
      DO 1 I=1,6
        IF (WR(I).LT.CHECK) GO TO 1
        CHECK=WR(I)
```

```
        IJ=I
1     CONTINUE
      OMEGA=DABS(WI(IJ))
      IF (WI(IJ).GT.0.D0) IJK=IJ




      IF (WI(IJ).LT.0.D0) IJK=IJ-1
      RETURN
      END
C=================================================================
      SUBROUTINE NORMAL(T)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION T(6,6),TNOR(6,6)
      CFAC=T(1,1)**2+T(1,2)**2
      IF (DABS(CFAC).LE.(1.D-10)) STOP 4001
      TNOR(1,1)=1.D0
      TNOR(2,1)=(T(1,1)*T(2,1)+T(2,2)*T(1,2))/CFAC
      TNOR(3,1)=(T(1,1)*T(3,1)+T(3,2)*T(1,2))/CFAC
      TNOR(4,1)=(T(1,1)*T(4,1)+T(4,2)*T(1,2))/CFAC
      TNOR(5,1)=(T(1,1)*T(5,1)+T(5,2)*T(1,2))/CFAC
      TNOR(6,1)=(T(1,1)*T(6,1)+T(6,2)*T(1,2))/CFAC
      TNOR(1,2)=0.D0
      TNOR(2,2)=(T(2,2)*T(1,1)-T(2,1)*T(1,2))/CFAC
      TNOR(3,2)=(T(3,2)*T(1,1)-T(3,1)*T(1,2))/CFAC
      TNOR(4,2)=(T(4,2)*T(1,1)-T(4,1)*T(1,2))/CFAC
      TNOR(5,2)=(T(5,2)*T(1,1)-T(5,1)*T(1,2))/CFAC
      TNOR(6,2)=(T(6,2)*T(1,1)-T(6,1)*T(1,2))/CFAC
      DO 1 I=1,6
        DO 2 J=1,2
          T(I,J)=TNOR(I,J)
2       CONTINUE
1     CONTINUE
      RETURN
      END
C=================================================================
      SUBROUTINE MULT(TINV,A,T,A2)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION TINV(6,6),A(6,6),T(6,6),A1(6,6),A2(6,6)
      DO 1 I=1,6
     DO 2 J=1,6
       A1(I,J)=0.D0
       A2(I,J)=0.D0
2    CONTINUE
1     CONTINUE
      DO 3 I=1,6
     DO 4 J=1,6
       DO 5 K=1,6
         A1(I,J)=A(I,K)*T(K,J)+A1(I,J)
```

```
5          CONTINUE
4        CONTINUE
3         CONTINUE
          DO 6 I=1,6
        DO 7 J=1,6
           DO 8 K=1,6
             A2(I,J)=TINV(I,K)*A1(K,J)+A2(I,J)
8          CONTINUE



7        CONTINUE
6         CONTINUE
          DO 11 I=1,6
C        WRITE (*,101)  (A(I,J),J=1,6)
11        CONTINUE
          DO 12 I=1,6
C        WRITE (*,101)  (T(I,J),J=1,6)
12        CONTINUE
          DO 10 I=1,6
C        WRITE (*,101)  (A2(I,J),J=1,6)
10        CONTINUE
C        WRITE (*,101) A2(1,1)
          RETURN
101       FORMAT (4E15.5)
          END
```

APPENDIX B

CRITICAL VALUE OF C FOR $[X, \hat{X}]$ BASIS

```
      PROGRAM NCCRIT
C     HOPF BIFURCATIONS
C     NOMOTO'S FIRST ORDER MODEL
C
C     CALCULATIONS FOR  CCRITICAL
C
C234567
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DOUBLE PRECISION K1,K2,K,LPSI,LY,LR,IZ,L,
     &       MASS,NV,NR,NVDOT,NRDOT,NDRS,NDRB,KPSI,KR,KY,K3
C
      DIMENSION AMAT(6,6),FV1(6),IV1(6),YYY(6,6)
      DIMENSION WR(6),WI(6),TSAVE(6,6),TLUD(6,6),IVLUD(6),SVLUD(6)
      DIMENSION ASAVE(6,6),A1(6,6),A2(6,6)
      OPEN (11,FILE='CVWN1.RES',STATUS='NEW')
      OPEN (12,FILE='CVWN2.RES',STATUS='NEW')
      OPEN (13,FILE='CVWN3.RES',STATUS='NEW')
      WRITE (*,*) 'ENTER MIN,MAX, AND INCREMENTS IN CCRIT'
      READ  (*,*)  CMIN,CMAX,IC
      WRITE (*,*) 'ENTER MIN,MAX, AND INCREMENTS IN WN'
      READ  (*,*)  WNMIN,WNMAX,INCR
      WRITE (*,*) 'ENTER WNO'
      READ  (*,*)  WNO
      WEIGHT=435.0
      IZ    =45.0
      L     =7.3
      RHO   =1.94
      G     =32.2
      XG    =0.0104
      MASS  =WEIGHT/G
      MASS  =MASS/(0.5*RHO*L**3)
      IZ    =IZ/(0.5*RHO*L**5)
      XG    =XG/L
      YRDOT =-0.00000
      YVDOT =-0.03430
      YR    =+0.00000
      YV    =-0.10700
      YDRS  =+0.01241
      YDRB  =+0.01241
      NRDOT =-0.00047
      NVDOT =-0.00000
      NR    =-0.00390
      NV    =-0.00000
```

```
      NDRS   =-0.337*YDRS
      NDRB   =+0.283*YDRS
      DH   =(IZ-NRDOT)*(MASS-YVDOT)-
     &       (MASS*XG-YRDOT)*(MASS*XG-NVDOT)
      A11=((IZ-NRDOT)*YV-(MASS*XG-YRDOT)*NV)/DH


      A12=((IZ-NRDOT)*(-MASS+YR)-
     &       (MASS*XG-YRDOT)*(-MASS*XG+NR))/DH
      A21=((MASS-YVDOT)*NV-(MASS*XG-NVDOT)*YV)/DH
      A22=((MASS-YVDOT)*(-MASS*XG+NR)-
     &       (MASS*XG-NVDOT)*(-MASS+YR))/DH
      B11=((IZ-NRDOT)*YDRS-(MASS*XG-YRDOT)*NDRS)/DH
      B12=((IZ-NRDOT)*YDRB-(MASS*XG-YRDOT)*NDRB)/DH
      B21=((MASS-YVDOT)*NDRS-(MASS*XG-NVDOT)*YDRS)/DH
      B22=((MASS-YVDOT)*NDRB-(MASS*XG-NVDOT)*YDRB)/DH
      B1 =B11-B12
      B2 =B21-B22
      C1=(A11*A22-A21*A12)*(A21*B1-A11*B2)
      C2=(A11+A22)*(A21*B1-A11*B2)+B2*(A11*A22-A21*A12)
      C3=-(A21*B1-A11*B2)**2
      A=C1/C2
      B=C3/C2
C
      EPS=1.D-5
      ILMAX=1500
C
      DO 1 II=1,INCR
C
      WN   =WNMIN+(WNMAX-WNMIN)*(II-1)/(INCR-1)
C      print *,wn
      ALPHA0=WN**3
      ALPHA1=2.15*WN**2
      ALPHA2=1.75*WN
      KPSI=-ALPHA1/B
      KY   =-ALPHA0/B
      KR   =-(ALPHA2+A)/B
      GAMA0=WNO**3
      GAMA1=2.15*WNO**2
      GAMA2=1.75*WNO
      LY=A+GAMA2
      LPSI=A*LY+GAMA1
      LR=A*LPSI+GAMA0
C234567
      DO 2 J=1,IC
      CCRIT=CMIN+(J-1)*(CMAX-CMIN)/(IC-1)
C     A    MATRIX
      AMAT(1,1)=0.0
      AMAT(1,2)=1.0
      AMAT(1,3)=0.0
```

```
      AMAT(1,4)=0.0
      AMAT(1,5)=0.0
      AMAT(1,6)=0.0
      AMAT(2,1)=0.0
      AMAT(2,2)=A
      AMAT(2,3)=0.0
      AMAT(2,4)=B*CCRIT*KPSI
      AMAT(2,5)=B*KR
      AMAT(2,6)=B*KY
      AMAT(3,1)=1.0
      AMAT(3,2)=0.0
      AMAT(3,3)=0.0
      AMAT(3,4)=0.0
      AMAT(3,5)=0.0
      AMAT(3,6)=0.0
      AMAT(4,1)=0.0
      AMAT(4,2)=0.0
      AMAT(4,3)=LPSI
      AMAT(4,4)=0.0
      AMAT(4,5)=1.0
      AMAT(4,6)=-LPSI
      AMAT(5,1)=0.0
      AMAT(5,2)=0.0
      AMAT(5,3)=LR
      AMAT(5,4)=B*CCRIT*KPSI
      AMAT(5,5)=B*KR
      AMAT(5,6)=-LR+B*KY
      AMAT(6,1)=0.0
      AMAT(6,2)=0.0
      AMAT(6,3)=LY
      AMAT(6,4)=1.0
      AMAT(6,5)=0.0
      AMAT(6,6)=-LY
C
      CALL RG(6,6,AMAT,WR,WI,0,ZZZ,IV1,FV1,IERR)
      CALL DSTABL(DEOS,WR,WI,FREQ)
      U=CCRIT
      IF (J.GT.1) GO TO 10
      DEOSOO=DEOS
      UOO=U
      LL=0
      GO TO 2
10    DEOSNN=DEOS
      UNN=U
      PR=DEOSNN*DEOSOO
      IF (PR.GT.0.D0) GO TO 3
      LL=LL+1
      IF (LL.GT.3) STOP 1000
      IL=0
      UO=UOO
```

```
      UN=UNN
      DEOSO=DEOSOO
      DEOSN=DEOSNN
6     UL=UO
      UR=UN
      DEOSL=DEOSO
      DEOSR=DEOSN
      U=(UL+UR)/2.D0
      CCRIT=U
      AMAT(1,1)=0.0
      AMAT(1,2)=1.0
      AMAT(1,3)=0.0
      AMAT(1,4)=0.0
      AMAT(1,5)=0.0
      AMAT(1,6)=0.0
      AMAT(2,1)=0.0
      AMAT(2,2)=A
      AMAT(2,3)=0.0
      AMAT(2,4)=B*CCRIT*KPSI
      AMAT(2,5)=B*KR
      AMAT(2,6)=B*KY
      AMAT(3,1)=1.0
      AMAT(3,2)=0.0
      AMAT(3,3)=0.0
      AMAT(3,4)=0.0
      AMAT(3,5)=0.0
      AMAT(3,6)=0.0
      AMAT(4,1)=0.0
      AMAT(4,2)=0.0
      AMAT(4,3)=LPSI
      AMAT(4,4)=0.0
      AMAT(4,5)=1.0
      AMAT(4,6)=-LPSI
      AMAT(5,1)=0.0
      AMAT(5,2)=0.0
      AMAT(5,3)=LR
      AMAT(5,4)=B*CCRIT*KPSI
      AMAT(5,5)=B*KR
      AMAT(5,6)=-LR+B*KY
      AMAT(6,1)=0.0
      AMAT(6,2)=0.0
      AMAT(6,3)=LY
      AMAT(6,4)=1.0
      AMAT(6,5)=0.0
      AMAT(6,6)=-LY
C
      CALL RG(6,6,AMAT,WR,WI,0,ZZZ,IV1,FV1,IERR)
      CALL DSTABL(DEOS,WR,WI,FREQ)
      U=CCRIT
      DEOSM=DEOS
```

```
      UM=U
      PRL=DEOSL*DEOSM
      PRR=DEOSR*DEOSM
      IF (PRL.GT.0.D0) GO TO 5
      UO=UL
      UN=UM
      DEOSO=DEOSL
      DEOSN=DEOSM
      IL=IL+1
      IF (IL.GT.ILMAX) STOP 3100
      DIF=DABS(UL-UM)
      IF (DIF.GT.EPS) GO TO 6
      U=UM
      GO TO 4
5     IF (PRR.GT.0.D0) STOP 3200
      UO=UM
      UN=UR
      DEOSO=DEOSM
      DEOSN=DEOSR
      IL=IL+1
      IF (IL.GT.ILMAX) STOP 3100
      DIF=DABS(UM-UR)
      IF (DIF.GT.EPS) GO TO 6
      U=UM
4     LLL=10+LL
      CCRIT=U
      WRITE (LLL,*) CCRIT,WN
3     UOO=UNN
      DEOSOO=DEOSNN
2     CONTINUE
1     CONTINUE
C
2001  FORMAT (2I5)
      END
C=======================================================
      SUBROUTINE DSTABL(DEOS,WR,WI,OMEGA)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION WR(6),WI(6)
      DEOS=-1.0D+20
      DO 1 I=1,6
        IF (WR(I).LT.DEOS) GO TO 1
        DEOS=WR(I)
        IJ=I
1     CONTINUE
      OMEGA=WI(IJ)
      OMEGA=DABS(OMEGA)
      RETURN
      END
C=======================================================
```

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APPENDIX C

HOPF BIFURCATION PROGRAM FOR $[X, \hat{X}]$ BASIS

```
      PROGRAM KKPSI
C       HOPF BIFURCATIONS
C       NOMOTO'S FIRST ORDER MODEL
C
C       CALCULATIONS FOR K AND CCRITICAL IF KPSI CHANGES W/ C
C
C234567
        IMPLICIT DOUBLE PRECISION (A-H,O-Z)
C
        DOUBLE PRECISION K1,K2,K,LPSI,LY,LR,IZ,L,
     &              MASS,NV,NR,NVDOT,NRDOT,NDRS,NDRB,KPSI,KR,KY,K3,
     &              L21,L22,L23,L24,L31,L32,L33,L34,L51,L52,L53,L54,
     &              M11,M12,M13,M14,M15,M16,M21,M22,M23,M24,M25,M26,
     &              M31,M32,M33,M34,M35,M36,M41,M42,M43,M44,M45,M46,
     &              M51,M52,M53,M54,M55,M56,M61,M62,M63,M64,M65,M66,
     &              N11,N12,N13,N14,N15,N16,N21,N22,N23,N24,N25,N26,
     &              N31,N32,N33,N34,N35,N36,N41,N42,N43,N44,N45,N46,
     &              N51,N52,N53,N54,N55,N56,N61,N62,N63,N64,N65,N66
C
        DIMENSION AMAT(6,6),T(6,6),TINV(6,6),FV1(6),IV1(6),YYY(6,6)
        DIMENSION WR(6),WI(6),TSAVE(6,6),TLUD(6,6),IVLUD(6),SVLUD(6)
        DIMENSION ASAVE(6,6),A1(6,6),A2(6,6)
        OPEN (10,FILE='CVWN1.RES',STATUS='OLD')
        OPEN (11,FILE='AKPSI.MAT',STATUS='NEW')
C        OPEN (12,FILE='IREAL.MAT',STATUS='NEW')
C        OPEN (13,FILE='RVALS.MAT',STATUS='NEW')
C        OPEN (14,FILE='KKKKK.MAT',STATUS='NEW')
        WEIGHT=435.0
        IZ     =45.0
        L      =7.3
        RHO    =1.94
        G      =32.2
        XG     =0.0104
        MASS   =WEIGHT/G
        MASS   =MASS/(0.5*RHO*L**3)
        IZ     =IZ/(0.5*RHO*L**5)
        XG     =XG/L
        YRDOT  =-0.00000
        YVDOT  =-0.03430
        YR     =+0.00000
        YV     =-0.10700
        YDRS   =+0.01241
        YDRB   =+0.01241
```

```
      NRDOT =-0.00047
      NVDOT =-0.00000
      NR    =-0.00390
      NV    =-0.00000
      NDRS  =-0.337*YDRS
      NDRB  =+0.283*YDRS
      DH   =(IZ-NRDOT)*(MASS-YVDOT)-
     &      (MASS*XG-YRDOT)*(MASS*XG-NVDOT)
      A11=((IZ-NRDOT)*YV-(MASS*XG-YRDOT)*NV)/DH
      A12=((IZ-NRDOT)*(-MASS+YR)-
     &      (MASS*XG-YRDOT)*(-MASS*XG+NR))/DH
      A21=((MASS-YVDOT)*NV-(MASS*XG-NVDOT)*YV)/DH
      A22=((MASS-YVDOT)*(-MASS*XG+NR)-
     &      (MASS*XG-NVDOT)*(-MASS+YR))/DH
      B11=((IZ-NRDOT)*YDRS-(MASS*XG-YRDOT)*NDRS)/DH
      B12=((IZ-NRDOT)*YDRB-(MASS*XG-YRDOT)*NDRB)/DH
      B21=((MASS-YVDOT)*NDRS-(MASS*XG-NVDOT)*YDRS)/DH
      B22=((MASS-YVDOT)*NDRB-(MASS*XG-NVDOT)*YDRB)/DH
      B1 =B11-B12
      B2 =B21-B22
C
  200 WRITE (*,1004)
      READ  (*,*)  IWN
      INCR=IWN
      WRITE (*,*) 'ENTER OBSERVER WN'
      READ (*,*) WNO
  205 WRITE(*,1007)
      READ (*,*) A3
C     D0 is Dsat

   50 WRITE (*,1006)
      READ  (*,*)    D0
      C1=(A11*A22-A21*A12)*(A21*B1-A11*B2)
      C2=(A11+A22)*(A21*B1-A11*B2)+B2*(A11*A22-A21*A12)
      C3=-(A21*B1-A11*B2)**2
      A=C1/C2
      B=C3/C2
C     A3=0.0
C
C
      DO 1 II=1,INCR
C
C
      READ (10,*) CCRIT,WN
C      print *,wn
      ALPHA0=WN**3
      ALPHA1=2.15*WN**2
      ALPHA2=1.75*WN
      KPSI=-ALPHA1/B
      KY  =-ALPHA0/B
```

```
      KR   =-(ALPHA2+A)/B
      GAMA0=WNO**3
      GAMA1=2.15*WNO**2
      GAMA2=1.75*WNO
      LY=A+GAMA2
      LPSI=A*LY+GAMA1
      LR=A*LPSI+GAMA0
C234567
C    A     MATRIX
      AMAT(1,1)=0.0
      AMAT(1,2)=1.0
      AMAT(1,3)=0.0
      AMAT(1,4)=0.0
      AMAT(1,5)=0.0
      AMAT(1,6)=0.0
      AMAT(2,1)=0.0
      AMAT(2,2)=A
      AMAT(2,3)=0.0
      AMAT(2,4)=B*CCRIT*KPSI
      AMAT(2,5)=B*KR
      AMAT(2   )=B*KY
      AMAT(3    =1.0
      AMAT(3, )=0.0
      AMAT(3,3)=0.0
      AMAT(3,4)=0.0
      AMAT(3,5)=0.0
      AMAT(3,6)=0.0
      AMAT(4,1)=0.0
      AMAT(4,2)=0.0
      AMAT(4,3)=LPSI
      AMAT(4,4)=0.0
      AMAT(4,5)=1.0
      AMAT(4,6)=-LPSI
      AMAT(5,1)=0.0
      AMAT(5,2)=0.0
      AMAT(5,3)=LR
      AMAT(5,4)=B*CCRIT*KPSI
      AMAT(5,5)=B*KR
      AMAT(5,6)=-LR+B*KY
      AMAT(6,1)=0.0
      AMAT(6,2)=0.0
      AMAT(6,3)=LY
      AMAT(6,4)=1.0
      AMAT(6,5)=0.0
      AMAT(6,6)=-LY
      DO 11 I=1,6
        DO 12 J=1,6
          ASAVE(I,J)=AMAT(I,J)
12      CONTINUE
11    CONTINUE
```

```
      CALL RG(6,6,AMAT,WR,WI,1,YYY,IV1,FV1,IERR)
      CALL DSOMEG(IEV,WR,WI,OMEGA,CHECK)
C      WRITE (*,*) IEV
      WRITE (12,*) (WR(IREAL),IREAL=1,6)
      OMEGA0=OMEGA
      DO 5 I=1,6
        T(I,1)=YYY(I,IEV)
        T(I,2)=-YYY(I,IEV+1)
5     CONTINUE
      IF(IEV.EQ.1) GO TO 13
      IF(IEV.EQ.2) GO TO 14
      IF(IEV.EQ.3) GO TO 15
      IF(IEV.EQ.4) GO TO 16
      IF(IEV.EQ.5) GO TO 17
      STOP 3004
13    DO 21 I=1,6
        T(I,3)=YYY(I,3)
        T(I,4)=YYY(I,4)
        T(I,5)=YYY(I,5)
        T(I,6)=YYY(I,6)
21    CONTINUE
      GO TO 30
14    DO 22 I=1,6
        T(I,3)=YYY(I,1)
        T(I,4)=YYY(I,4)
        T(I,5)=YYY(I,5)
        T(I,6)=YYY(I,6)
22    CONTINUE
      GO TO 30
15    DO 23 I=1,6
        T(I,3)=YYY(I,1)
        T(I,4)=YYY(I,2)
        T(I,5)=YYY(I,5)
        T(I,6)=YYY(I,6)
23    CONTINUE
      GO TO 30
16    DO 24 I=1,6
        T(I,3)=YYY(I,1)
        T(I,4)=YYY(I,2)
        T(I,5)=YYY(I,3)
        T(I,6)=YYY(I,6)
24    CONTINUE
      GO TO 30
17    DO 25 I=1,6
        T(I,3)=YYY(I,1)
        T(I,4)=YYY(I,2)
        T(I,5)=YYY(I,3)
        T(I,6)=YYY(I,4)
25    CONTINUE
30    CONTINUE
```

```
C
C     NORMALIZATION OF THE CRITICAL EIGENVECTOR
C
      CALL NORMAL(T)
C
C     INVERT TRANSFORMATION MATRIX
C
      DO 2 I=1,6
        DO 3 J=1,6
          TINV(I,J)=0.0
          TSAVE(I,J)=T(I,J)
3       CONTINUE
2     CONTINUE
      CALL DLUD(6,6,TSAVE,6,TLUD,IVLUD)
      DO 4 I=1,6
        IF (IVLUD(I).EQ.0) STOP 3003
4     CONTINUE
      CALL DILU(6,6,TLUD,IVLUD,SVLUD)
      DO 8 I=1,6
        DO 9 J=1,6
          TINV(I,J)=TLUD(I,J)
9     CONTINUE
8     CONTINUE
C
C     CHECK  Inv(T)*A*T
C
      IMULT=1
      IF (IMULT.EQ.1) CALL MULT(TINV,ASAVE,T,A2)
      IF (IMULT.EQ.0) STOP
      P=A2(3,3)
      Q=A2(4,4)
      R=A2(5,5)
      S=A2(6,6)
C      WRITE(21,*) P,Q,R,S

C     DEFINITION OF Nij
C
      N11=TINV(1,1)
      N21=TINV(2,1)
      N31=TINV(3,1)
      N41=TINV(4,1)
      N51=TINV(5,1)
      N61=TINV(6,1)
      N12=TINV(1,2)
      N22=TINV(2,2)
      N32=TINV(3,2)
      N42=TINV(4,2)
      N52=TINV(5,2)
      N62=TINV(6,2)
      N13=TINV(1,3)
```

```
      N23=TINV(2,3)
      N33=TINV(3,3)
      N43=TINV(4,3)
      N53=TINV(5,3)
      N63=TINV(6,3)
      N14=TINV(1,4)
      N24=TINV(2,4)
      N34=TINV(3,4)
      N44=TINV(4,4)
      N54=TINV(5,4)
      N64=TINV(6,4)
      N15=TINV(1,5)
      N25=TINV(2,5)
      N35=TINV(3,5)
      N45=TINV(4,5)
      N55=TINV(5,5)
      N65=TINV(6,5)
      N16=TINV(1,6)
      N26=TINV(2,6)
      N36=TINV(3,6)
      N46=TINV(4,6)
      N56=TINV(5,6)
      N66=TINV(6,6)
C
C     DEFINITION OF Mij
C
      M11=T(1,1)
      M21=T(2,1)
      M31=T(3,1)
      M41=T(4,1)
      M51=T(5,1)
      M61=T(6,1)
      M12=T(1,2)
      M22=T(2,2)
      M32=T(3,2)
      M42=T(4,2)
      M52=T(5,2)
      M62=T(6,2)
      M13=T(1,3)
      M23=T(2,3)
      M33=T(3,3)
      M43=T(4,3)
      M53=T(5,3)
      M63=T(6,3)
      M14=T(1,4)
      M24=T(2,4)
      M34=T(3,4)
      M44=T(4,4)
      M54=T(5,4)
      M64=T(6,4)
```

```
      M15=T(1,5)
      M25=T(2,5)
      M35=T(3,5)
      M45=T(4,5)
      M55=T(5,5)
      M65=T(6,5)
      M16=T(1,6)
      M26=T(2,6)
      M36=T(3,6)
      M46=T(4,6)
      M56=T(5,6)
      M66=T(6,6)
C
C      K1=1./8.*((ALPHA2**3)+ALPHA0)/(ALPHA2)
C      K2=3.*A3-.5*(ALPHA2**2)/ALPHA0
C      K3=1./((B**2)*(D0**2))*(ALPHA2+A)*((ALPHA0/ALPHA2)+(A**2))
C      K=K1*(K2+K3)
C
C      print *, wn,k,ccrit
C
      BL=B/(3*D0**2)
C
C23456789012345678901234567890123456789012345678901234567890123456789012345
      L21=A3*M21**3-BL*(CCRIT**3*KPSI**3*M41**3+KR**3*M51**3+
     &      KY**3*M61**3+
     &      3*CCRIT**2*KPSI**2*KR*M41**2*M51+
     &      3*CCRIT**2*KPSI**2*KY*M41**2*M61+
     & 3*CCRIT*KPSI*KR**2*M51**2*M41+3*CCRIT*KPSI*KY**2*M61**2*M41+
     &    3*KR*KY**2*M61**2*M51+3*KR**2*KY*M51**2*M61+
     &    6*CCRIT*KPSI*KR*KY*M41*M51*M61)
      L22=A3*M22**3-BL*(CCRIT**3*KPSI**3*M42**3+KR**3*M52**3+
     &      KY**3*M62**3+
     &      3*CCRIT**2*KPSI**2*KR*M42**2*M52+
     &      3*CCRIT**2*KPSI**2*KY*M42**2*M62+
     & 3*CCRIT*KPSI*KR**2*M52**2*M42+3*CCRIT*KPSI*KY**2*M62**2*M42+
     &    3*KR*KY**2*M62**2*M52+3*KR**2*KY*M52**2*M62+
     &    6*CCRIT*KPSI*KR*KY*M42*M52*M62)
      L23=3*A3*M21**2*M22-BL*(3*CCRIT**3*KPSI**3*M41**2*M42+
     &      3*KR**3*M51**2*M52+
     &      3*KY**3*M61**2*M62+
     &      3*CCRIT**2*KPSI**2*KR*(M41**2*M52+2*M41*M42*M51)+
     &    3*CCRIT**2*KPSI**2*KY*(M41**2*M62+2*M41*M42*M61)+
     &    3*CCRIT*KPSI*KR**2*(M51**2*M42+2*M51*M52*M41)+
     &    3*CCRIT*KPSI*KY**2*(M61**2*M42+2*M61*M62*M41)+
     &    3*KR*KY**2*(M61**2*M52+2*M61*M62*M51)+
     &    3*KR**2*KY*(M51**2*M62+2*M51*M52*M61)+
     &   6*CCRIT*KPSI**KR*KY*(M41*M51*M62+(M41*M52+M42*M51)*M61))
      L24=3*A3*M21*M22**2-BL*(3*CCRIT**3*KPSI**3*M41*M42**2+
     &      3*KR**3*M51*M52**2+
     &      3*KY**3*M61*M62**2+
```

```
     &        3*CCRIT**2*KPSI**2*KR*(M42**2*M51+2*M41*M42*M52)+
     &     3*CCRIT**2*KPSI**2*KY*(M42**2*M61+2*M41*M42*M62)+
     &     3*CCRIT*KPSI*KR**2*(M52**2*M41+2*M51*M52*M42)+
     &        3*CCRIT*KPSI*KY**2*(M62**2*M41+2*M61*M62*M42)+
     &     3*KR*KY**2*(M62**2*M51+2*M61*M62*M52)+
     &     3*KR**2*KY*(M52**2*M61+2*M51*M52*M62)+
     &     6*CCRIT*KPSI*KR*KY*(M42*M52*M61+(M41*M52+M42*M51)*M62))
      L31=(-1/6)*M11**3
      L32=(-1/6)*M12**3
      L33=(-1/2)*M11**2*M12
      L34=(-1/2)*M11*M12**2
C2345678901234567890123456789012345678901234567890123456789012345
      L51=-BL*(CCRIT**3*KPSI**3*M41**3+KR**3*M51**3+KY**3*M61**3+
     &        3*CCRIT**2*KPSI**2*KR*M41**2*M51+
     &        3*CCRIT**2*KPSI**2*KY*M41**2*M61+
     & 3*CCRIT*KPSI*KR**2*M51**2*M41+3*CCRIT*KPSI*KY**2*M61**2*M41+
     &        3*KR*KY**2*M61**2*M51+3*KR**2*KY*M51**2*M61+
     &     6*CCRIT*KPSI*KR*KY*M41*M51*M61)
      L52=-BL*(CCRIT**3*KPSI**3*M42**3+KR**3*M52**3+KY**3*M62**3+
     &        3*CCRIT**2*KPSI**2*KR*M42**2*M52+
     &        3*CCRIT**2*KPSI**2*KY*M42**2*M62+
     & 3*CCRIT*KPSI*KR**2*M52**2*M42+3*CCRIT*KPSI*KY**2*M62**2*M42+
     &     3*KR*KY**2*M62**2*M52+3*KR**2*KY*M52**2*M62+
     &     6*CCRIT*KPSI*KR*KY*M42*M52*M62)
      L53=-BL*(3*CCRIT**3*KPSI**3*M41**2*M42+3*KR**3*M51**2*M52+
     &        3*KY**3*M61**2*M62+
     &        3*CCRIT**2*KPSI**2*KR*(M41**2*M52+2*M41*M42*M51)+
     &     3*CCRIT**2*KPSI**2*KY*(M41**2*M62+2*M41*M42*M61)+
     &        3*CCRIT*KPSI*KR**2*(M51**2*M42+2*M51*M52*M41)+
     &        3*CCRIT*KPSI*KY**2*(M61**2*M42+2*M61*M62*M41)+
     &     3*KR*KY**2*(M61**2*M52+2*M61*M62*M51)+
     &     3*KR**2*KY*(M51**2*M62+2*M51*M52*M61)+
     &     6*CCRIT*KPSI*KR*KY*(M41*M51*M62+(M41*M52+M42*M51)*M61))
      L54=-BL*(3*CCRIT**3*KPSI**3*M41*M42**2+3*KR**3*M51*M52**2+
     &        3*KY**3*M61*M62**2+
     &        3*CCRIT**2*KPSI**2*KR*(M42**2*M51+2*M41*M42*M52)+
     &     3*CCRIT**2*KPSI**2*KY*(M42**2*M61+2*M41*M42*M62)+
     &     3*CCRIT*KPSI*KR**2*(M52**2*M41+2*M51*M52*M42)+
     &        3*CCRIT*KPSI*KY**2*(M62**2*M41+2*M61*M62*M42)+
     &     3*KR*KY**2*(M62**2*M51+2+M61*M62*M52)+
     &     3*KR**2*KY*(M52**2*M61+2*M51*M52*M62)+
     &     6*CCRIT*KPSI*KR*KY*(M42*M52*M61+(M41*M52+M42*M51)*M62))
      R11=N12*L21+N13*L31+N15*L51
      R12=N12*L23+N13*L33+N15*L53
      R13=N12*L24+N13*L34+N15*L54
      R14=N12*L22+N13*L32+N15*L52
      R21=N22*L21+N23*L31+N25*L51
      R22=N22*L23+N23*L33+N25*L53
      R23=N22*L24+N23*L34+N25*L54
      R24=N22*L22+N23*L32+N25*L52
```

```
C
      WRITE (13,*) R11,R13,R22,R24
      K=(3*R11+R13+R22+3*R24)/8
      WRITE (11,2001) WN,K,CCRIT
    1 CONTINUE
      STOP
 1004 FORMAT (' ENTER NUMBER OF DATA')
 1006 FORMAT (' ENTER DELTASAT.')
 1007 FORMAT ('ENTER A3')
 2001 FORMAT (3E15.5)
      END
C=================================================================
      SUBROUTINE DSOMEG(IJK,WR,WI,OMEGA,CHECK)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION WR(6),WI(4)
      CHECK=-1.0D+25
      DO 1 I=1,6
        IF (WR(I).LT.CHECK) GO TO 1
        CHECK=WR(I)
        IJ=I
1     CONTINUE
      OMEGA=DABS(WI(IJ))
      IF (WI(IJ).GT.0.D0) IJK=IJ
      IF (WI(IJ).LT.0.D0) IJK=IJ-1
      RETURN
      END
C=================================================================
      SUBROUTINE NORMAL(T)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION T(6,6),TNOR(6,6)
      CFAC=T(1,1)**2+T(1,2)**2
      IF (DABS(CFAC).LE.(1.D-10)) STOP 4001
      TNOR(1,1)=1.D0
      TNOR(2,1)=(T(1,1)*T(2,1)+T(2,2)*T(1,2))/CFAC
      TNOR(3,1)=(T(1,1)*T(3,1)+T(3,2)*T(1,2))/CFAC
      TNOR(4,1)=(T(1,1)*T(4,1)+T(4,2)*T(1,2))/CFAC
      TNOR(5,1)=(T(1,1)*T(5,1)+T(5,2)*T(1,2))/CFAC
      TNOR(6,1)=(T(1,1)*T(6,1)+T(6,2)*T(1,2))/CFAC
      TNOR(1,2)=0.D0
      TNOR(2,2)=(T(2,2)*T(1,1)-T(2,1)*T(1,2))/CFAC
      TNOR(3,2)=(T(3,2)*T(1,1)-T(3,1)*T(1,2))/CFAC
      TNOR(4,2)=(T(4,2)*T(1,1)-T(4,1)*T(1,2))/CFAC
      TNOR(5,2)=(T(5,2)*T(1,1)-T(5,1)*T(1,2))/CFAC
      TNOR(6,2)=(T(6,2)*T(1,1)-T(6,1)*T(1,2))/CFAC
      DO 1 I=1,6
        DO 2 J=1,2
          T(I,J)=TNOR(I,J)
2       CONTINUE
1     CONTINUE
      RETURN
```

```
      END
C=================================================================
      SUBROUTINE MULT(TINV,A,T,A2)
      IMPLICIT DOUBLE PRECISION (A-H,O-Z)
      DIMENSION TINV(6,6),A(6,6),T(6,6),A1(6,6),A2(6,6)
      DO 1 I=1,6
     DO 2 J=1,6
       A1(I,J)=0.D0
       A2(I,J)=0.D0
2    CONTINUE
1     CONTINUE
      DO 3 I=1,6
     DO 4 J=1,6
       DO 5 K=1,6
         A1(I,J)=A(I,K)*T(K,J)+A1(I,J)
5      CONTINUE
4    CONTINUE
3     CONTINUE
      DO 6 I=1,6
     DO 7 J=1,6
       DO 8 K=1,6
         A2(I,J)=TINV(I,K)*A1(K,J)+A2(I,J)
8      CONTINUE
7    CONTINUE
6     CONTINUE
      DO 11 I=1,6
C    WRITE (*,101) (A(I,J),J=1,6)
11    CONTINUE
      DO 12 I=1,6
C    WRITE (*,101) (T(I,J),J=1,6)
12    CONTINUE
      DO 10 I=1,6
C    WRITE (*,101) (A2(I,J),J=1,6)
10    CONTINUE
C    WRITE (*,101) A2(1,1)
      RETURN
101   FORMAT (4E15.5)
      END
```

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